Math 203 | Solution
Quiz 10 | 5 December 2003

1. Let C be the circle $x^2+y^2=4$, $z=0$, oriented counterclockwise seen from above. Let $\mathbf{F}=(2-y)\vec{\imath}+x\vec{\jmath}-x^2z\vec{k}$. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

   Solution: We parametrize C as $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 0\rangle$, $0\le t\le 2\pi$. We have

   $$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle.$$

   Next, we evaluate

   $$\mathbf{F}(\mathbf{r}(t)) = \langle 2-2\sin t, 2\cos t, 0\rangle$$

   Putting the integral together, we get

   $$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi} -4\sin t+4\sin^2 t+4\cos^2 t\,dt \\ &= 8\pi\end{aligned}$$

2. Let S be the part of the paraboloid $z=4-x^2-y^2$ with $z\ge 0$, oriented upward. Let $\mathbf{G}=2xz\vec{\jmath}+2\vec{k}$. Compute $\iint_S \mathbf{G}\cdot d\mathbf{S}$.

   Solution: We parametrize S as

   $$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 4-u^2\rangle$$

   with $0\le v\le 2\pi$ and $0\le u\le 2$. The direction of increasing u is raidally outward, and the direction of increasing v is counterclockwise, so that the orientation of this parametrization is upward. We get

   $$\begin{aligned}\mathbf{r}_u\times\mathbf{r}_v &= \langle\cos v, \sin v, -2u\rangle\times\langle -u\sin v, u\cos v, 0\rangle \\ &= \langle 2u^2\cos v, 2u^2\sin v, u\rangle\end{aligned}$$

   and

   $$\mathbf{G}(\mathbf{r}(u,v)) = \langle 0, 2u(4-u^2)\cos v, 2\rangle$$

   Setting up the integral, we get

   $$\begin{aligned}\iint_S \mathbf{G}\cdot d\mathbf{S} &= \int_0^2\int_0^{2\pi} 4u^3(4-u^2)\cos v\sin v+2u\,dv\,du \\ &= \int_0^2 4\pi u\,du \\ &= 8\pi\end{aligned}$$

   Alternatively, we *might* have noticed that $\mathbf{G}$ is the curl of the field $\mathbf{F}$ in problem 1, and that C in problem 1 is the boundary of S, so that

   $$\iint_S \mathbf{G}\cdot d\mathbf{S} = \iint_S \text{curl }\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = 8\pi$$