Math 203 | Solution
Quiz 5 | 17 October 2003

1. Let $f(x,y) = \dfrac{e^{-x^2}}{1+y^2}$. If $x = r\cos\theta$ and $y = r\sin\theta$, find $\dfrac{\partial f}{\partial \theta}$ when $r = 2$ and $\theta = \dfrac{\pi}{2}$.

   Solution: By the chain rule, we have

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\frac{2xe^{-x^2}}{1+y^2}(-r\sin\theta) - \frac{2ye^{-x^2}}{(1+y^2)^2}(r\cos\theta)
\end{aligned}$$

   When $r = 2$ and $\theta = \dfrac{\pi}{2}$, we have $x = 0$ and $y = 2$. The first term above is thus zero. But the second term is zero, too, because $r\cos\theta = 0$ when $\theta = \dfrac{\pi}{2}$. The answer is 0.

2. Let $f(x,y,z) = x^2y + yz$. Find the directional derivative of f in the direction of the vector $\vec{\imath} + \vec{k}$ at the point $(2,3,4)$.

   Solution: The unit vector in the given direction is $\vec{u} = \dfrac{1}{\sqrt{2}}(\vec{\imath} + \vec{k})$. We also need

$$\begin{aligned}
\nabla f &= 2xy\vec{\imath} + (x^2+z)\vec{\jmath} + y\vec{k} \\
\nabla f(2,3,4) &= 12\vec{\imath} + 8\vec{\jmath} + 3\vec{k}.
\end{aligned}$$

   Finally, we get

$$\begin{aligned}
D_{\vec{u}} f(2,3,4) &= \frac{1}{\sqrt{2}}(\vec{\imath} + \vec{k}) \cdot (12\vec{\imath} + 8\vec{\jmath} + 3\vec{k}) \\
&= \frac{15}{\sqrt{2}}.
\end{aligned}$$

3. Suppose f is a function of x and y, $x = t$ and $y = e^{-t}$. Fill in the blanks:

$$\frac{d^2 f}{dt^2} = \boxed{}\frac{\partial^2 f}{\partial x^2} + \boxed{}\frac{\partial^2 f}{\partial x\,\partial y} + \boxed{}\frac{\partial^2 f}{\partial y^2} + \boxed{}\frac{\partial f}{\partial x} + \boxed{}\frac{\partial f}{\partial y}.$$

Solution: We have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} - e^{-t}\frac{\partial f}{\partial y}.\end{aligned}$$

Differentiating again with respect to t, we get

$$\begin{aligned}\frac{d^2 f}{dt^2} &= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dt} - e^{-t}\left(\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right) + e^{-t}\frac{\partial f}{\partial y} \\ &= \frac{\partial^2 f}{\partial x^2} - e^{-t}\frac{\partial^2 f}{\partial x\,\partial y} - e^{-t}\left(\frac{\partial^2 f}{\partial x\,\partial y} - e^{-t}\frac{\partial^2 f}{\partial y^2}\right) + e^{-t}\frac{\partial f}{\partial y} \\ &= \frac{\partial^2 f}{\partial x^2} - 2e^{-t}\frac{\partial^2 f}{\partial x\,\partial y} + e^{-2t}\frac{\partial^2 f}{\partial y^2} + e^{-t}\frac{\partial f}{\partial y}.\end{aligned}$$