Math 203 — Solution

Quiz 6 — 24 October 2003

1. Find all critical points of the function $f(x, y) = x^2y^2 - 3x^2y + 2x^2 + y^2 - 5y$. Do not bother to classify them.

   Solution: We have

$$\begin{aligned} f_x &= 2xy^2 - 6xy + 4x \\ &= 2x(y^2 - 3y + 2) \\ &= 2x(y-1)(y-2) \\ f_y &= 2x^2y - 3x^2 + 2y - 5. \end{aligned}$$

   Setting $f_x = 0$, we get three cases:
   Case 1: $x = 0$. Then $2y - 5 = 0$, so $y = 5/2$.
   Case 2: $y = 1$. Then $-x^2 - 3 = 0$, so $x^2 = -3$. There is no solution.
   Case 3: $y = 2$. Then $x^2 - 1 = 0$, so $x = \pm 1$.
   There are three critical points. They are

$$\left(0, \frac{5}{2}\right), \quad (1, 2), \quad (-1, 2).$$

2. The function $f(x, y) = y^3 - 7y + y^2 + x^2 - 4x + 2xy$ has critical points at $(1, 1)$ and $(3, -1)$. Classify each as a local minimum, local maximum, or saddle point.

   Solution: We have

$$\begin{aligned} f_x &= 2x - 4 + 2y \\ f_y &= 3y^2 - 7 + 2y + 2x \\ f_{xx} &= 2 \\ f_{xy} &= 2 \\ f_{yy} &= 6y + 2 \end{aligned}$$

   Thus $D(x, y) = 12y + 4 - 4 = 12y$. At the critical point $(1, 1)$, $D > 0$ and $f_{xx} > 0$, so $(1, 1)$ is a local minimum. At the critical point $(3, -1)$, $D < 0$, so $(3, -1)$ is a saddle point.

3. Suppose f is a function of x and y, $x = t$ and $y = e^{3t}$. Fill in the blanks:

$$\frac{d^2 f}{dt^2} = \boxed{}\frac{\partial^2 f}{\partial x^2} + \boxed{}\frac{\partial^2 f}{\partial x\,\partial y} + \boxed{}\frac{\partial^2 f}{\partial y^2} + \boxed{}\frac{\partial f}{\partial x} + \boxed{}\frac{\partial f}{\partial y}.$$

Solution: We have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x} + 3e^{3t}\frac{\partial f}{\partial y}.\end{aligned}$$

Differentiating again with respect to t, we get

$$\begin{aligned}\frac{d^2 f}{dt^2} &= \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dt} + 3e^{3t}\left(\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right) + 9e^{3t}\frac{\partial f}{\partial y}\\ &= \frac{\partial^2 f}{\partial x^2} + 3e^{3t}\frac{\partial^2 f}{\partial x\,\partial y} + 3e^{3t}\left(\frac{\partial^2 f}{\partial x\,\partial y} + 3e^{3t}\frac{\partial^2 f}{\partial y^2}\right) + 9e^{3t}\frac{\partial f}{\partial y}\\ &= \frac{\partial^2 f}{\partial x^2} + 6e^{3t}\frac{\partial^2 f}{\partial x\,\partial y} + 9e^{6t}\frac{\partial^2 f}{\partial y^2} + 9e^{3t}\frac{\partial f}{\partial y}.\end{aligned}$$