Math 202 **Integration Example** **27 February 2002**

We evaluate the indefinite integral $\int \frac{1}{x^3+1}\,dx$.

To begin, we need the partial-fractions decomposition of $\frac{1}{x^3+1}$. We factor x^3+1 as

$$x^3+1 \quad = \quad (x+1)(x^2-x+1).$$

The second factor is a quadratic irreducible, so the partial-fractions expansion of $\frac{1}{x^3+1}$ has the form

$$\frac{1}{x^3+1} \quad = \quad \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Multiplying through by $(x+1)(x^2-x+1)$, we get

$$\begin{aligned} 1 &= A(x^2-x+1) + (Bx+C)(x+1) && (1)\\ &= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C && (2)\\ &= (A+B)x^2 + (-A+B+C)x + (A+C). && (3) \end{aligned}$$

Equating the coefficients on both sides of lines (1)-(3) we get the system

$$\begin{array}{rcrcrcl} A & + & B & & & = & 0\\ -A & + & B & + & C & = & 0\\ A & & & + & C & = & 1. \end{array}$$

The third equation says $C = 1 - A$, so we can substitute $1-A$ for C in the second equation to get (after some algebra) $2A - B = 1$. Adding this to the first equation gives $3A = 1$, so $A = \frac{1}{3}$. The first equation then implies that $B = -\frac{1}{3}$, and the third equation that $C = \frac{2}{3}$.

Thus we have

$$\frac{1}{x^3+1} \quad = \quad \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right],$$

so that

$$\int \frac{1}{x^3+1}\,dx \quad = \quad \frac{1}{3}\left[\int \frac{dx}{x+1} - \int \frac{x-2}{x^2-x+1}\,dx\right]. \qquad (4)$$

The first integral is easy: the substitution $u = x + 1$; $du = dx$ shows that the integral is a natural logarithm. We get

$$\int \frac{1}{x^3+1}\,dx \;=\; \frac{1}{3}\left[\ln|x+1| - \int \frac{x-2}{x^2-x+1}\,dx\right].$$

We begin working on the remaining integral by completing the square in the denominator. We have

$$x^2 - x + 1 \;=\; \left(x - \frac{1}{2}\right) + \frac{3}{4}.$$

We get

$$\int \frac{1}{x^3+1}\,dx \;=\; \frac{1}{3}\left[\ln|x+1| - \int \frac{x-2}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx\right].$$

Next we try the substitution $u = x - \dfrac{1}{2}$; $du = dx$. This implies that $x = u + \dfrac{1}{2}$. We get

$$\begin{aligned}\int \frac{1}{x^3+1}\,dx &= \frac{1}{3}\left[\ln|x+1| - \int \frac{u+\frac{1}{2}-2}{u^2+\frac{3}{4}}\,du\right]\\ &= \frac{1}{3}\left[\ln|x+1| - \int \frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}\,du\right].\end{aligned}$$

We can write the remaining integral as the sum of two integrals. We get

$$\int \frac{1}{x^3+1}\,dx \;=\; \frac{1}{3}\left[\ln|x+1| - \int \frac{u}{u^2+\frac{3}{4}}\,du + \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du\right].$$

For the first integral, we use the substitution $w = u^2 + \dfrac{3}{4}$; $dw = du$. For the second, we use the trig substitution

$$\begin{aligned}u &= \frac{\sqrt{3}}{2}\tan\theta\\ du &= \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta.\end{aligned}$$

Making these substitutions, we get

$$\begin{aligned}
\int \frac{1}{x^3+1}\,dx &= \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\int \frac{dw}{w} + \frac{3}{2}\int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}\tan^2\theta + \frac{3}{4}}\,d\theta\right] \\
&= \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\int \frac{dw}{w} + \frac{3\sqrt{3}}{4}\int \frac{\sec^2\theta}{\frac{3}{4}\sec^2\theta}d\theta\right] \\
&= \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\int \frac{dw}{w} + \sqrt{3}\int d\theta\right].
\end{aligned}$$

Now we can perform the integration. We get

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\ln|w| + \sqrt{3}\theta\right].$$

Finally, we unwind the substitutions. We have

$$w = u^2 + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = x^2 - x + 1$$

and

$$\theta = \tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

Making these substitutions, we get

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\ln|x^2-x+1| + \sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right] + C.$$

To check this, we differentiate our final answer. We get

$$\frac{d}{dx}\left(\frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\ln|x^2-x+1| + \sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]\right) =$$

$$\begin{aligned}
&\frac{1}{3}\left[\frac{1}{x+1} - \frac{1}{2}\left(\frac{2x-1}{x^2-x+1}\right) + \sqrt{3}\frac{\frac{2}{\sqrt{3}}}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\right] \\
= &\frac{1}{3}\left[\frac{1}{x+1} - \frac{x-\frac{1}{2}}{x^2-x+1} + \frac{2}{1+\frac{4x^2-4x+1}{3}}\right]
\end{aligned}$$

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\begin{aligned}
&= \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-\frac{1}{2}}{x^2-x+1} + \frac{6}{4x^2-4x+4}\right] \\
&= \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-\frac{1}{2}}{x^2-x+1} + \frac{\frac{3}{2}}{x^2-x+1}\right] \\
&= \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right] \\
&= \frac{1}{3}\left[\frac{(x^2-x+1)-(x-2)(x+1)}{(x+1)(x^2-x+1)}\right] \\
&= \frac{1}{3}\left[\frac{(x^2-x+1)-(x^2-x-2)}{x^3+1}\right] \\
&= \frac{1}{3}\left[\frac{3}{x^3+1}\right] \\
&= \frac{1}{x^3+1}.
\end{aligned}
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And there was much rejoicing.