Math 202(01) | Solution
Hour Exam I | 21 – 22 February 2002

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1. If $\tanh x = \dfrac{4}{5}$, find $\cosh x$ and $\sinh x$.

   Solution: We use the identity $1 - \tanh^2 x = \operatorname{sech}^2 x$ to get

$$\begin{aligned} \operatorname{sech}^2 x &= 1 - \frac{16}{25} \\ &= \frac{9}{25} \end{aligned}$$

   so that $\operatorname{sech} x = \dfrac{3}{5}$. Thus $\cosh x = \dfrac{5}{3}$. Then, since $\sinh x = \tanh x \cosh x$, we get

$$\begin{aligned} \sinh x &= \frac{4}{5} \times \frac{5}{3} \\ &= \frac{4}{3}. \end{aligned}$$

2. Find $\lim_{x \to 0} (1 + \sin 2x)^{\frac{1}{x}}$.

   Solution: The limit has the indeterminate form 1^{∞}. We set

$$y = \lim_{x \to 0} (1 + \sin 2x)^{\frac{1}{x}}$$

   and take logarithms to get

$$\ln y = \lim_{x \to 0} \frac{\ln(1 + \sin 2x)}{x}.$$

   The limit on the right has the form $0/0$, so we may apply l'Hospital's rule to get

$$\begin{aligned} \ln y &= \lim_{x \to 0} \frac{2 \cos 2x}{1 + \sin 2x} \\ &= 2 \end{aligned}$$

   Thus $y = e^2$.

3. Compute $\displaystyle\int_0^4 \frac{1}{\sqrt{9 - 2x}}\, dx$.

Solution: Let $u = 9 - 2x$ so that $du = -2\,dx$. We have

$$\begin{aligned} \int \frac{1}{\sqrt{5-2x}}\,dx &= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du \\ &= -\sqrt{u} + C \\ &= -\sqrt{9-2x} + C \end{aligned}$$

So that

$$\begin{aligned} \int_0^4 \sqrt{9-2x}\,dx &= \Big[-\sqrt{9-2x}\Big]_0^4 \\ &= -\sqrt{1} + \sqrt{9} \\ &= 2. \end{aligned}$$

4. Find $\displaystyle\int \frac{x^5}{1+x^3}\,dx$.

   Solution: Set $u = 1 + x^3$ so that $du = 3x^2\,dx$. We'll also need to use the fact that $x^3 = u - 1$. We transform the integral to

$$\begin{aligned} \int \frac{x^3}{1+x^3}\,x^2\,dx &= \frac{1}{3}\int \frac{u-1}{u}\,du \\ &= \frac{1}{3}\int 1 - \frac{1}{u}\,du \\ &= \frac{1}{3}\left(u - \ln|u|\right) + C \\ &= \frac{1}{3}\Big((1+x^3) - \ln|1+x^3|\Big) + C. \end{aligned}$$

5. Find $\displaystyle\int x^2 \sin 2x\,dx$

   Solution: We integrate by parts. Set

$$\begin{aligned} u &= x^2 & v &= -\frac{\cos 2x}{2} \\ du &= 2x\,dx & dv &= \sin 2x\,dx. \end{aligned}$$

   We get

$$\int x^2 \cos 2x\,dx = -\frac{x^2\cos 2x}{2} + \int x\cos 2x\,dx.$$

We need to evaluate $\int x \cos 2x \, dx$. We do this by parts as well. Set

$$\begin{aligned} u &= x & v &= \frac{\sin 2x}{2} \\ du &= dx & dv &= \cos 2x \, dx. \end{aligned}$$

Thus

$$\begin{aligned} \int x \cos 2x \, dx &= \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx \\ &= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C \end{aligned}$$

Substituting this into our original integral, we get

$$\begin{aligned} \int x^2 \cos 2x \, dx &= -\frac{x^2 \cos 2x}{2} + \int x \cos 2x \, dx \\ &= -\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C. \end{aligned}$$

6. Find $\int \cos^5 x \, dx$.

Solution: Write

$$\begin{aligned} \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 \cos x \, dx \\ &= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx. \end{aligned}$$

Set $u = \sin x$ so that $du = \cos x \, dx$. We get

$$\begin{aligned} \int \cos^5 x \, dx &= \int (1 - 2u^2 + u^4) \, du \\ &= u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\ &= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C. \end{aligned}$$

7. Use a trig substitution to eliminate the square root from the integral $\int \sqrt{9+4x^2}\,dx$. Do not try to evaluate the integral; just rewrite it as an integral $d\theta$ with no square root.

   Solution: We want $4x^2 = 9\tan^2\theta$. We can write this as $2x = 3\tan\theta$, so we'll set

$$\begin{aligned} x &= \frac{3}{2}\tan\theta \\ dx &= \frac{3}{2}\sec^2\theta\,d\theta. \end{aligned}$$

   The subsitution yields

$$\begin{aligned} \int \sqrt{9+4x^2}\,dx &= \int \sqrt{9+9\tan^2\theta}\,\frac{3}{2}\sec^2\theta\,d\theta \\ &= \frac{9}{2}\int \sec^3\theta\,d\theta. \end{aligned}$$

8. Compute $\int \frac{dx}{x\sqrt{x^2-9}}$.

   Solution: We use the trig substitution $x = 3\sec\theta$ so that $dx = 3\sec\theta\tan\theta\,d\theta$. Making the substitution, we get

$$\begin{aligned} \int \frac{dx}{x\sqrt{x^2-9}} &= \int \frac{3\sec\theta\tan\theta}{3\sec\theta\sqrt{9\sec^2\theta-9}}\,d\theta \\ &= \frac{1}{3}\int d\theta \\ &= \frac{\theta}{3} + C \\ &= \frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C. \end{aligned}$$