Math 202(01) | Solution
Hour Exam II | 28 – 29 March 2002

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1. Compute $\displaystyle\int \frac{5x^2-7x-1}{(x^2+1)(x-4)}\,dx$.

Solution: We use partial fractions. We have

$$\frac{5x^2-7x-1}{(x^2+1)(x-4)} = \frac{Ax+B}{x^2+1}+\frac{C}{x-4}.$$

Multiplying through to clear denominators, we get

$$\begin{aligned}5x^2-7x-1 &= (Ax+B)(x-4)+C(x^2+1)\\ &= (A+C)x^2+(-4A+B)x+(-4B+C)\end{aligned}$$

We get the system

$$\begin{array}{rcrcrcr} A & & & + & C & = & 5\\ -4A & + & B & & & = & -7\\ & - & 4B & + & C & = & -1\end{array}$$

The calculator says the solution is $A=2$, $B=1$, and $C=3$. We get

$$\begin{aligned}\int \frac{5x^2-7x-1}{(x^2+1)(x-4)}\,dx &= \int \frac{2x}{x^2+1}+\frac{1}{x^2+1}+\frac{3}{x-4}\,dx\\ &= \ln(x^2+1)+\tan^{-1}(x)+3\ln|x-4|+C.\end{aligned}$$

2. Compute $\displaystyle\int \frac{2x^3+5x^2-3x-1}{2x-1}\,dx$.

Solution: The degree of the numerator is greater than the degree of the denominator, so we begin by dividing. We have

$$\begin{array}{r|rcrcrcr} & x^2 & + & 3x & - & \frac{1}{2x-1} & & \\ \hline 2x-1 & 2x^3 & + & 5x^2 & - & 3x & - & 1\\ & 2x^3 & - & x^2 & & & & \\ \hline & & & 6x^2 & - & 3x & & \\ & & & 6x^2 & - & 3x & & \\ \hline & & & & & 0 & - & 1\end{array}$$

Thus

$$\int \frac{2x^3+5x^2-3x-2}{2x-1}\,dx = \int x^2+3x-\frac{1}{2x-1}\,dx$$
$$= \frac{x^3}{3}+\frac{3x^2}{2}-\frac{1}{2}\ln|2x-1|+C.$$

3. Consider $\int_1^3 \ln x\,dx$.

(a) Estimate the value of the integral using Simpson's rule with $n=4$ sub-intervals. Round your answer to six decimal places.

Solution: Let $f(x)=\ln x$. We have

$$\begin{aligned} S_4 &= \frac{1}{6}\left[f(1)+4f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+4f\left(\frac{5}{2}\right)+f(3)\right] \\ &= \frac{1}{6}\left[\ln(1)+4\ln(3/2)+2\ln(2)+4\ln(5/2)+\ln(3)\right]. \end{aligned}$$

This is easy enough to approximate with a calculator; we get

$$S_4 \approx 1.295322.$$

(b) Find an upper bound for the absolute value of the error in your approximation.

Solution: We'll need K_4. We have

$$\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \\ f^{(4)}(x) &= -\frac{6}{x^4}. \end{aligned}$$

For $1\le x\le 3$, the maximum value of $|f^{(4)}(x)|$ is 6, so we may take $K_4=6$. Our error bound is

$$E_S \le \frac{6(3-1)^5}{180(4)^4}$$

$$\begin{aligned} &= \frac{6 \times 32}{180 \times 256} \\ &= \frac{1}{240} \\ &\approx 0.00416667. \end{aligned}$$

(c) What is the smallest value of n we may use in Simpson's rule to guarantee an error of less than 10^{-6}?

Solution: We need to solve the inequality

$$\frac{6 \times 32}{180n^4} < 10^{-6}$$

which simplifies to

$$\begin{aligned} \frac{16}{15n^4} &< 10^{-6} \\ \frac{16 \times 10^6}{15} &< n^4, \end{aligned}$$

or $n > \sqrt[4]{\frac{16 \times 10^6}{15}}$. The number on the right is approximately 32.137. We need to take the next-greatest even integer, so we take $n = 34$ to achieve the desired accuracy.

4. Rewrite the integral $\int_5^\infty \frac{dx}{x^2 + x}$ as a limit, and find its value.

Solution: The integral is improper, and we rewrite it as

$$\int_5^\infty \frac{dx}{x^2 + x} = \lim_{t \to \infty} \int_5^t \frac{dx}{x^2 + x}.$$

To find its value, we use partial fractions. We have

$$\frac{1}{x^2 + x} = \frac{A}{x} + \frac{B}{x + 1}.$$

Multiplying through to clear denominators, we get

$$1 = A(x + 1) + Bx,$$

which gives us $A = 1$ and $B = -1$. Thus

$$\begin{aligned} \int \frac{dx}{x^2 + x} &= \int \frac{1}{x} - \frac{1}{x + 1}\, dx \\ &= \ln|x| - \ln|x + 1|. \end{aligned}$$

We have

$$\begin{aligned}
\int_5^\infty \frac{dx}{x^2+x} &= \lim_{t\to\infty} [\ln t - \ln(t+1) - \ln(5) + \ln(6)] \\
&= \ln\frac{6}{5} + \lim_{t\to\infty} \ln\left(\frac{t}{t+1}\right) \\
&= \ln\frac{6}{5} + \ln\left(\lim_{t\to\infty} \frac{t}{t+1}\right) \\
&= \ln\frac{6}{5} + \ln 1 \\
&= \ln\frac{6}{5}.
\end{aligned}$$

5. Evaluate the definite integral $\displaystyle\int_0^4 \frac{1}{(x-3)^2}\,dx$.

Solution: We notice that the integrand has a vertical asymptote at $x = 3$, and since this lies in the interval over which we are integrating, we will need to treat this as an improper integral. We have

$$\begin{aligned}
\int_0^4 \frac{1}{(x-3)^2}\,dx &= \int_0^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx \\
&= \lim_{t\to 3^-} \int_0^t \frac{dx}{(x-3)^2} + \lim_{t\to 3^+} \int_t^4 \frac{dx}{(x-3)^2}.
\end{aligned}$$

Carrying out the first integration, we get

$$\begin{aligned}
\lim_{t\to 3^-} \int_0^t \frac{dx}{(x-3)^2} &= \lim_{t\to 3^-} \left[-\frac{1}{x-3}\right]_0^t \\
&= \lim_{t\to 3^-} \left(-\frac{1}{t-3}\right) + \frac{1}{3} \\
&= \lim_{t\to 3^-} \left(\frac{1}{3-t}\right) + \frac{1}{3}.
\end{aligned}$$

As $t \to 3^-$, the fraction $\displaystyle\frac{1}{3-t}$ goes to positive infinity. Thus the first of the two integrals is divergent, and so the entire integral is divergent.