Math 202(01)	Solution
Hour Exam III	25 – 26 April 2002

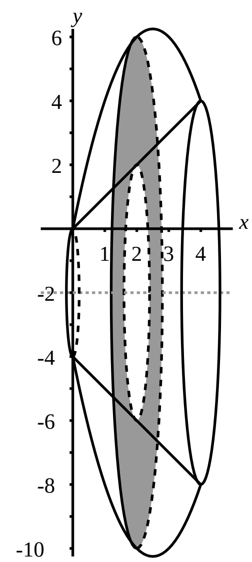
1. (a) Let R be the region bounded by the parabola $y = 5x - x^2$ and the line $y = x$. Set up, but do not evaluate, an integral for the volume generated when R is revolved about the line $y = -2$.

Solution: The given parabola and line intersect at the origin and the point $(4, 4)$. We use the washers method. The washer at position x has

thickness dx
outer radius $5x - x^2 - (-2)$
inner radius $x - (-2)$.

The volume is given by

$$\begin{aligned} V &= \int_0^4 \pi((5x - x^2 - (-2))^2 - (x - (-2))^2)\, dx \\ &= \int_0^4 \pi((5x - x^2 + 2)^2 - (x + 2)^2)\, dx. \end{aligned}$$



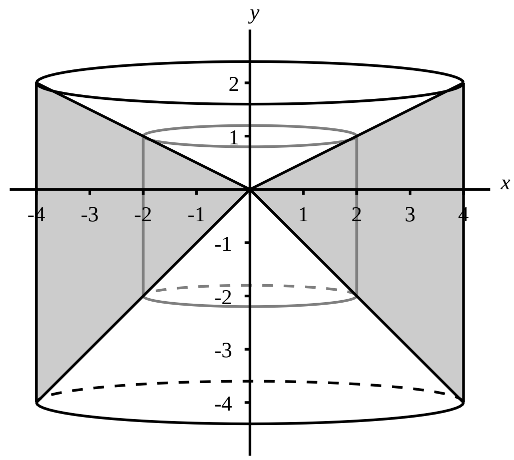
(b) Let R be the triangular region bounded by the lines $y = \dfrac{x}{2}$, $y = -x$, and $x = 4$. Set up, but do not evaluate, an integral for the volume generated when R is revolved about the y-axis.

Solution: The washer method would be difficult, so we try the shell method. The shell at position x has

thickness dx
radius x
height $\dfrac{x}{2} - (-x)$.

The volume is given by

$$V = \int_0^4 2\pi x \left(\frac{x}{2} + x\right) dx.$$



2. Set up, but do not evaluate, an integral for the area of the surface generated when the part of the curve $y = \sin x$ with $0 \le x \le \pi$ is revolved about the line $y = -3$.

Solution: The band at position x has radius $3 + \sin x$ and slant height $\sqrt{1 + \cos^2 x}\, dx$. The surface area is given by

$$S = \int_0^\pi (3 + \sin x)\sqrt{1 + \cos^2 x}\, dx.$$

3. (a) Consider the sequence $\{a_n\}$ given by $a_n = \dfrac{n}{\ln n}$ (for $n \ge 2$). Find $\lim\limits_{n\to\infty} a_n$.

Solution: Let $f(x) = \dfrac{x}{\ln x}$. Then $\lim\limits_{x\to\infty} f(x)$ has the form ∞/∞, so we can apply l'Hospital's rule to get

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1}{1/x} \\ &= \lim_{x\to\infty} x \\ &= \infty. \end{aligned}$$

The sequence has the same limit: $\lim\limits_{n\to\infty} \dfrac{n}{\ln n} = \infty$.

(b) Consider the sequence $\{a_n\}$ given by $a_n = (-1)^n \cos(n\pi)$. Find $\lim_{n\to\infty} a_n$.

Solution: The first few terms of the sequence are

$$\begin{array}{rcccl} a_1 & = & (-1)\cos\pi & = & 1 \\ a_2 & = & \cos 2\pi & = & 1 \\ a_3 & = & (-1)\cos 3\pi & = & 1 \\ a_4 & = & \cos 4\pi & = & 1. \end{array}$$

We see a pattern developing here – every term of the sequence is 1. We deduce that

$$\lim_{n\to\infty} (-1)^{n-1} \cos n\pi = 1.$$

4. Determine whether each series converges or diverges. Give reasons, and show all necessary work.

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n-\sqrt{n}}$

Solution: For $n \geq 4$, we know that $n - \sqrt{n} < n$, so that

$$\frac{1}{n-\sqrt{n}} > \frac{1}{n}.$$

We also know that $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$ is divergent, so we conclude that the given series is divergent by the Basic Comparison Test.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Solution: Since both x and $\ln x$ are positive and increase as x increases, we know that the function $\dfrac{1}{x\ln x}$ is positive and decreasing, so we can use the integral test. We evaluate

$$\begin{aligned} \int_2^{\infty} \frac{dx}{x\ln x} &= \lim_{t\to\infty} \int_2^t \frac{dx}{x\ln x} \\ &= \lim_{t\to\infty} [\ln\ln x]_2^t \\ &= \lim_{t\to\infty} (\ln\ln t - \ln\ln 2) \\ &= \infty \end{aligned}$$

Since the integral is divergent, so also is the series.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3+2^{-n}}$

Solution: Since

$$\lim_{n\to\infty} \frac{1}{3+2^{-n}} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{2^n}} = \frac{1}{3+0} \neq 0$$

this series diverges by the Divergence Test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{n^2+3}$

Solution: This is an alternating series, so we apply the alternating series test. The absolute values of the terms are given by

$$a_n = \frac{n}{n^2+3}$$

We first verify that $\lim_{n\to\infty} a_n = 0$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n^2+3} &= \lim_{n\to\infty} \frac{1/n}{1+3/n^2} \\
&= \frac{0}{1+0} \\
&= 0.
\end{aligned}$$

Next, we need to verify that a_n is a decreasing sequence. There are a few ways to do this; here's one: Let $f(x) = \dfrac{x}{x^2+3}$. Then

$$\begin{aligned}
f'(x) &= \frac{x^2+3-2x^2}{(x^2+3)^2} \\
&= 3-x^2.
\end{aligned}$$

For $x \geq 2$, the derivative is negative. Thus the sequence of absolute values is eventually decreasing. By the alternating series test, then, the original series converges.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{2n+1}{n^3-1}$

Solution: Let $a_n = \dfrac{2n+1}{n^3-1}$ and let $b_n = \dfrac{1}{n^2}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n+1}{n^3-1} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{2n^3+n^2}{n^3-1} \\
&= 2.
\end{aligned}$$

Since 2 is finite and non-zero, we can use the Limit Comparison Theorem, along with the fact that $\sum \frac{1}{n^2}$ converges (it is a p-series with $p > 1$) to conclude that the given series is convergent.

5. Find the sum $\sum_{n=1}^{\infty} \frac{(-3)^n}{4^{n+1}}$.

   Solution: The first few terms of the series are

$$-\frac{3}{4^2} + \frac{3^2}{4^3} - \frac{3^3}{4^4} + \cdots$$

   The series is geometric with $a = -\frac{3}{16}$ and $r = -\frac{3}{4}$. Thus the sum is given by

$$\begin{aligned} \frac{a}{1-r} &= -\frac{3}{16} \times \frac{1}{1+\frac{3}{4}} \\ &= -\frac{3}{16} \times \frac{4}{7} \\ &= -\frac{3}{28}. \end{aligned}$$