1. (a) Let $R$ be the region bounded by the parabola $y = 5x - x^2$ and the line $y = x$. Set up, but do not evaluate, an integral for the volume generated when $R$ is revolved about the line $y = -2$.

Solution: The given parabola and line intersect at the origin and the point $(4, 4)$. We use the washers method. The washer at position $x$ has

- thickness $dx$
- outer radius $5x - x^2 - (-2)$
- inner radius $x - (-2)$.

The volume is given by

$$V = \int_{0}^{4} \pi((5x - x^2 - (-2))^2 - (x - (-2))^2) \, dx$$

$$= \int_{0}^{4} \pi((5x - x^2 + 2)^2 - (x + 2)^2) \, dx.$$
(b) Let \( R \) be the triangular region bounded by the lines \( y = \frac{x}{2} \), \( y = -x \), and \( x = 4 \).

Set up, but do not evaluate, an integral for the volume generated when \( R \) is revolved about the \( y \)-axis.

Solution: The washer method would be difficult, so we try the shell method. The shell at position \( x \) has

- thickness \( dx \)
- radius \( x \)
- height \( \frac{x}{2} - (-x) \).

The volume is given by

\[
V = \int_{0}^{4} 2\pi x \left( \frac{x}{2} + x \right) \, dx.
\]

2. Set up, but do not evaluate, an integral for the area of the surface generated when the part of the curve \( y = \sin x \) with \( 0 \leq x \leq \pi \) is revolved about the line \( y = -3 \).

Solution: The band at position \( x \) has radius \( 3 + \sin x \) and slant height \( \sqrt{1 + \cos^2 x} \, dx \).

The surface area is given by

\[
S = \int_{0}^{\pi} (3 + \sin x) \sqrt{1 + \cos^2 x} \, dx.
\]

3. (a) Consider the sequence \( \{a_n\} \) given by \( a_n = \frac{n}{\ln n} \) (for \( n \geq 2 \)). Find \( \lim_{n \to \infty} a_n \).

Solution: Let \( f(x) = \frac{x}{\ln x} \). Then \( \lim_{x \to \infty} f(x) \) has the form \( \infty/\infty \), so we can apply l'Hospital’s rule to get

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1/x} = \lim_{x \to \infty} x = \infty.
\]

The sequence has the same limit: \( \lim_{n \to \infty} \frac{n}{\ln n} = \infty \).
(b) Consider the sequence \( \{a_n\} \) given by \( a_n = (-1)^n \cos(n\pi) \). Find \( \lim_{n \to \infty} a_n \).

Solution: The first few terms of the sequence are

\[
\begin{align*}
a_1 &= (-1) \cos \pi = 1 \\
a_2 &= \cos 2\pi = 1 \\
a_3 &= (-1) \cos 3\pi = 1 \\
a_4 &= \cos 4\pi = 1.
\end{align*}
\]

We see a pattern developing here – every term of the sequence is 1. We deduce that

\[ \lim_{n \to \infty} (-1)^n \cos n\pi = 1. \]

4. Determine whether each series converges or diverges. Give reasons, and show all necessary work.

(a) \( \sum_{n=4}^{\infty} \frac{1}{n - \sqrt{n}} \)

Solution: For \( n \geq 4 \), we know that \( n - \sqrt{n} < n \), so that

\[ \frac{1}{n - \sqrt{n}} > \frac{1}{n}. \]

We also know that \( \sum_{n=4}^{\infty} \frac{1}{n} \) is divergent, so we conclude that the given series is divergent by the Basic Comparison Test.

(b) \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

Solution: Since both \( x \) and \( \ln x \) are positive and increase as \( x \) increases, we know that the function \( \frac{1}{x \ln x} \) is positive and decreasing, so we can use the integral test. We evaluate

\[
\int_{2}^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_{2}^{t} \frac{dx}{x \ln x} = \lim_{t \to \infty} [\ln \ln x]_{2}^{t} = \lim_{t \to \infty} (\ln \ln t - \ln \ln 2) = \infty.
\]

Since the integral is divergent, so also is the series.
(c) \( \sum_{n=0}^{\infty} \frac{1}{3 + 2^{-n}} \)
Solution: Since
\[ \lim_{n \to \infty} \frac{1}{3 + 2^{-n}} = \lim_{n \to \infty} \frac{1}{3 + \frac{1}{2^n}} = \frac{1}{3 + 0} \neq 0 \]
this series diverges by the Divergence Test.

(d) \( \sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{n^2 + 3} \)
Solution: This is an alternating series, so we apply the alternating series test. The absolute values of the terms are given by
\[ a_n = \frac{n}{n^2 + 3} \]
We first verify that \( \lim_{n \to \infty} a_n = 0 \). We have
\[ \lim_{n \to \infty} \frac{n}{n^2 + 3} = \lim_{n \to \infty} \frac{1/n}{1 + 3/n^2} \]
\[ = \frac{1}{1 + 0} = 0. \]
Next, we need to verify that \( a_n \) is a decreasing sequence. There are a few ways to do this; here's one: Let \( f(x) = \frac{x}{x^2 + 3} \). Then
\[ f'(x) = \frac{x^2 + 3 - 2x^2}{(x^2 + 3)^2} \]
\[ = 3 - x^2. \]
For \( x \geq 2 \), the derivative is negative. Thus the sequence of absolute values is eventually decreasing. By the alternating series test, then, the original series converges.

(e) \( \sum_{n=2}^{\infty} \frac{2n + 1}{n^3 - 1} \)
Solution: Let \( a_n = \frac{2n + 1}{n^3 - 1} \) and let \( b_n = \frac{1}{n^2} \). Then
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2n + 1}{n^3 - 1} \cdot \frac{n^2}{1} \]
\[ = \lim_{n \to \infty} \frac{2n^3 + n^2}{n^3 - 1} \]
\[ = 2. \]
Since 2 is finite and non-zero, we can use the Limit Comparison Theorem, along with the fact that \( \sum \frac{1}{n^2} \) converges (it is a \( p \)-series with \( p > 1 \)) to conclude that the given series is convergent.

5. Find the sum \( \sum_{n=1}^{\infty} \frac{(-3)^n}{4^{n+1}} \).

Solution: The first few terms of the series are

\[
-\frac{3}{4^2} + \frac{3^2}{4^3} - \frac{3^3}{4^4} + \cdots
\]

The series is geometric with \( a = -\frac{3}{16} \) and \( r = -\frac{3}{4} \). Thus the sum is given by

\[
\frac{a}{1-r} = -\frac{3}{16} \times \frac{1}{1 + \frac{3}{4}}
\]

\[
= -\frac{3}{16} \times \frac{4}{7}
\]

\[
= -\frac{3}{28}.
\]