Math 202(01) Solution
Final Exam May 2002

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1. Find $\lim_{x\to 0}(1+\sin(\pi x))^{\frac{1}{x}}$.

   Solution: Let

$$y = \lim_{x\to 0}(1+\sin(\pi x))^{\frac{1}{x}}.$$

   Then

$$\begin{aligned}
\ln y &= \lim_{xto0} \frac{\ln(1+\sin(\pi x))}{x} \\
&= \lim_{x\to 0} \frac{\pi\cos(\pi x)}{1+\sin(\pi x)} \\
&= \pi
\end{aligned}$$

   by l'Hospital's rule. Thus $y = e^{\pi}$.

2. Compute $\int x \ln x\, dx$

   Solution: Let

$$\begin{aligned}
u &= \ln x & v &= x^2/2 \\
du &= 1/x\, dx & dv &= x\, dx
\end{aligned}$$

   Then

$$\begin{aligned}
\int x \ln x\, dx &= \frac{x^2 \ln x}{2} - \int \frac{x}{2}\, dx \\
&= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.
\end{aligned}$$

3. (a) Find the correct partial-fractions decomposition of $\dfrac{6x^2-5x+6}{(x^2+1)(x-2)}$.

   Do not integrate.

   Solution: We write

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-2} = \frac{6x^2-5x+6}{(x^2+1)(x-2)}$$

and clear denominators to get

$$\begin{aligned}6x^2 - 5x + 6 &= (Ax + B)(x - 2) + C(x^2 + 1) \\ &= (A + C)x^2 + (B - 2A)x + (C - 2B).\end{aligned}$$

This gives us the system

$$\begin{array}{rcrcrcr} A & & & + & C & = & 6 \\ -2A & + & B & & & = & -5 \\ & - & 2B & + & C & = & 6 \end{array}$$

The solution is $A = 2$, $B = -1$, and $C = 4$. We have

$$\frac{6x^2 - 5x + 6}{(x^2 + 1)(x - 2)} = \frac{2x - 1}{x^2 + 1} + \frac{4}{x - 2}.$$

(b) Use a trigonometric substitution to simplify $\displaystyle\int \frac{dx}{(x^2 + 4x)^{\frac{3}{2}}}$.

Do not integrate; just rewrite the integral as a $d\theta$ integral with no fractional powers or square roots.

Solution: We write

$$x^2 + 4x = (x + 2)^2 - 4$$

showing that the substitution we want is

$$\begin{aligned}x + 2 &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta\, d\theta.\end{aligned}$$

The integral becomes

$$\begin{aligned}\int \frac{2\sec\theta\tan\theta}{(4\sec^2\theta - 4)^{\frac{3}{2}}}\, d\theta &= \int \frac{2\sec\theta\tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}}\, d\theta \\ &= \int \frac{2\sec\theta\tan\theta}{8\tan^3\theta}\, d\theta \\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta.\end{aligned}$$

4. (a) Use Simpson's rule with $n = 8$ sub-intervals to estimate $\displaystyle\int_1^5 \frac{dx}{x}$.

Solution: The division points are

$$1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5.$$

We have $\Delta x = \frac{1}{2}$. Simpson's rule says that the integral is approximated by

$$\frac{\Delta x}{3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right]$$

That is,

$$\begin{aligned}\int_1^5 \frac{dx}{x} &\approx \frac{1}{6}\left[1 + \frac{8}{3} + \frac{2}{2} + \frac{8}{5} + \frac{2}{3} + \frac{8}{7} + \frac{2}{4} + \frac{8}{9} + \frac{1}{5}\right] \\ &= \frac{6089}{3780} \\ &\approx 1.610847.\end{aligned}$$

(b) Find an upper bound on the error in your approximation in part 4a.

Solution: We find that $|f^{(4)}(x)| = \frac{24}{x^5}$. The maximum value of $|f^{(4)}(x)|$ on the interval $[1, 5]$ is

$$|f^{(4)}(1)| = 24.$$

Thus we may take $K_4 = 24$. We have

$$\begin{aligned}|E_S| &\leq \frac{24 \times (5-1)^5}{180 \times 8^4} \\ &= \frac{4 \times 2^{10}}{30 \times 2^{12}} \\ &= \frac{1}{30}.\end{aligned}$$

5. Determine whether each of the following series converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2-1}{2n^3}$

Solution: We first test for absolute convergence. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{2n^3}$ is a positive-term series. We use the limit comparison theorem with $\sum \frac{1}{n}$. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2-1}{2n^3} \cdot \frac{n}{1} &= \lim_{n\to\infty} \frac{n^3-n}{2n^3} \\ &= \frac{1}{2}.\end{aligned}$$

Since this is finite and positive, we know that the series $\sum_{n=1}^{\infty} \frac{n^2-1}{2n^3}$ behaves like $\sum \frac{1}{n}$. The latter is the harmonic series, which is known to be divergent. Therefore our given series does not converge absolutely.
To test for conditional convergence, we first note that

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2-1}{2n^3} &= \lim_{n\to\infty} \frac{1/n-1/n^3}{2} \\ &= 0.\end{aligned}$$

We check to see that the absolute values of the terms form a decreasing sequence as follows. Let

$$f(x) = \frac{x^2-1}{2x^3}.$$

Then

$$\begin{aligned}f'(x) &= \frac{4x^4-6x^2(x^2-1)}{4x^6} \\ &= \frac{2x^2(3-x^2)}{4x^6}.\end{aligned}$$

When $x > \sqrt{3}$, the numerator (and thus the whole fraction) is negative. Since the function f decreases for $x > \sqrt{3}$, we know that the absolute values of the terms in the series decrease for $n \geq 2$.
Thus by the Alternating Series Test, the original series is conditionally convergent.

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{5n+2^{-n}}$
Solution: We note that

$$\lim_{n\to\infty} \frac{n}{5n+2^{-n}} = \frac{1}{5}.$$

Since the absolute values of the terms do not approach zero, the terms themselves do not approach zero, and so the series is divergent by the Divergence Test.

6. Find the interval of convergence for the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{n2^n}$.

Solution: We apply the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{|x-1|^n} &= \lim_{n\to\infty} \frac{n}{2(n+1)}|x-1| \\ &= \frac{|x-1|}{2}.\end{aligned}$$

The series converges absolutely when $\dfrac{|x-1|}{2} < 1$. That is, when $|x-1| < 2$, or

$$\begin{array}{rcccl} -2 & < & x-1 & < & 2 \\ -1 & < & x & < & 3 \end{array}$$

We check the endpoints. At $x = -1$, we have

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{n2^n} \quad = \quad \sum_{n=1}^{\infty} \frac{2^n}{n2^n}.$$

This is the harmonic series, and it diverges. At $x = 3$, we have

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n2^n} \quad = \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

This is the alternating harmonic series, and it converges. The interval of convergence is

$$-1 < x \leq 3.$$

7. Find a power series representation for the function $f(x) = \dfrac{1}{(x+3)^2}$.

Indicate the interval on which your power series representation is valid.

Solution: We differentiate both sides of

$$\sum_{n=0}^{\infty} x^n \quad = \quad \frac{1}{1-x}$$

to get

$$\sum_{n=1}^{\infty} nx^{n-1} \quad = \quad \frac{1}{(1-x)^2}.$$

Next we substitute $-\dfrac{x}{3}$ for x to get

$$\frac{1}{(1+\frac{x}{3})^2} \quad = \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^{n-1}}{3^{n-1}}.$$

Finally we multiply both sides by $\dfrac{1}{9}$ to get

$$\frac{1}{(3+x)^2} \quad = \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nx^{n-1}}{3^{n+1}}.$$

The series is valid for $\left|\dfrac{x}{3}\right| < 1$; that is, for $-3 < x < 3$.

8. Use a power series to estimate the value of $\int_0^{\frac{1}{3}} \frac{dx}{1+x^5}$ with an error of less than 10^{-6}.

Solution: We have

$$\begin{aligned}\frac{1}{1+x^5} &= \sum_{n=0}^{\infty}(-x^5)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{5n}\end{aligned}$$

which is valid for $|x| < 1$. Thus

$$\begin{aligned}\int_0^{\frac{1}{3}} \frac{dx}{1+x^5} &= \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{5n+1}}{5n+1}\right]_0^{\frac{1}{3}} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(5n+1)\cdot 3^{5n+1}} \\ &= \frac{1}{1\cdot 3^1} - \frac{1}{6\cdot 3^6} + \frac{1}{11\cdot 3^{11}} - \cdots\end{aligned}$$

Since $\frac{1}{11\cdot 3^{11}} < 10^{-6}$, we know the sum of the first two terms has the required accuracy:

$$\begin{aligned}\int_0^{\frac{1}{3}} \frac{dx}{1+x^5} &\approx \frac{1}{3} - \frac{1}{6\times 3^6} \\ &\approx 0.333105\end{aligned}$$

9. Find the Taylor series for $\ln x$ about 1.

Write down the first six terms and the general form of the n^{th} term.

Solution: We have

$$\begin{aligned}f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= 1/x & f'(1) &= 1 \\ f''(x) &= -1/x^2 & f''(1) &= -1 \\ f'''(x) &= 2/x^3 & f'''(1) &= 2 \\ f^{(4)}(x) &= -6/x^4 & f^{(4)}(1) &= -6 \\ f^{(5)}(x) &= 24/x^5 & f^{(5)}(1) &= 24\end{aligned}$$

and in general,

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n \qquad f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

The Taylor series begins

$$
\begin{aligned}
T(x) &= (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \cdots \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \cdots
\end{aligned}
$$

The n^{th} term of the series is

$$
(-1)^{n-1}\frac{(n-1)!(x-1)^n}{n!} = (-1)^{n-1}\frac{(x-1)^n}{n}.
$$