Math 202 **Festival of Integration** **February 2002**

Problem: Compute $\int \frac{1}{u^2(u^2-1)^2}\,du$

Solution: The denominator factors as $u^2(u+1)^2(u-1)^2$. The partial-fractions decomposition form is

$$\frac{1}{u^2(u^2-1)^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2} + \frac{E}{u-1} + \frac{F}{(u-1)^2}.$$

Clearing denominators, we get

$$\begin{aligned}(A+C+E)u^5 + (B-C+D+E+F)u^4 + (-2A-C-2D-E+2F)u^3 \\ +(-2B+C+D-E+F)u^2 + Au + B = 1.\end{aligned}$$

From this, we immediately get $A = 0$ and $B = 1$. The remaining variables must satisfy the system

$$\begin{gathered} C+E=0, \quad 1-C+D+E+F=0, \\ -C-2D-E+2F=0, \quad -2+C+D-E+F=0 \end{gathered}$$

This is easily solved either by inspection or on a calculator. The solution is

$$C = \frac{3}{4}, \quad D = \frac{1}{4}, \quad E = -\frac{3}{4}, \quad F = \frac{1}{4}.$$

We can now complete the integral. We get $\int \frac{1}{u^2(u^2-1)^2}\,du =$

$$\begin{aligned} & \int \frac{1}{u^2} + \frac{3}{4(u+1)} + \frac{1}{4(u+1)^2} - \frac{3}{4(u-1)} + \frac{1}{4(u-1)^2}\,du \\ = \; & -\frac{1}{u} + \frac{3}{4}\ln|u+1| - \frac{1}{4(u+1)} - \frac{3}{4}\ln|u-1| - \frac{1}{4(u-1)} + C \\ = \; & -\frac{1}{u} + \frac{3}{4}\ln\left|\frac{u+1}{u-1}\right| + \frac{u}{2(u^2-1)} + C. \end{aligned}$$

Problem: Compute $\int e^{2x} \sin 3x\,dx$.

Solution: We use integration by parts, taking

$$\begin{aligned} u &= e^{2x} & v &= -\frac{1}{3}\cos 3x \\ du &= 2e^{2x}\,dx & dv &= \sin 3x\,dx \end{aligned}$$

We get

$$\int e^{2x}\sin 3x\,dx = -\frac{e^{2x}\cos 3x}{3} + \frac{2}{3}\boxed{\int e^{2x}\cos 3x\,dx}.$$

We use integration by parts on the expression in the box, taking

$$\begin{aligned} u &= e^{2x} & v &= \frac{1}{3}\sin 3x \\ du &= 2e^{2x}\,dx & dv &= \cos 3x\,dx \end{aligned}$$

We get

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}\sin 3x}{3} - \frac{2}{3}\int e^{2x}\sin 3x\,dx.$$

Substituting this back into the box, we get

$$\begin{aligned} \int e^{2x}\sin 3x\,dx &= -\frac{e^{2x}\cos 3x}{3} + \frac{2}{3}\left[\frac{e^{2x}\sin 3x}{3} - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right] \\ &= -\frac{e^{2x}\cos 3x}{3} + \frac{2}{3}\left(\frac{e^{2x}\sin 3x}{3}\right) - \frac{4}{9}\int e^{2x}\sin 3x\,dx. \end{aligned}$$

We now add $\frac{4}{9}\int e^{2x}\sin 3x\,dx$ to each side of the integral above to get

$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{e^{2x}\cos 3x}{3} + \frac{2}{3}\left(\frac{e^{2x}\sin 3x}{3}\right).$$

Finally, we solve for the unknown, getting

$$\int e^{2x}\sin 3x\,dx = -\frac{3}{13}(e^{2x}\cos 3x) + \frac{2}{13}(e^{2x}\sin 3x) + C.$$

Problem: Compute $\int \frac{1}{1-x^4}\,dx$

Solution: We factor the denominator as

$$1 - x^4 \quad = \quad (1-x)(1+x)(1+x^2).$$

The form of the partial-fractions decomposition for this function is

$$\frac{1}{1-x^4} \quad = \quad \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}.$$

Clearing denominators, we get

$$(A-B-C)x^3 + (A+B-D)x^2 + (A-B+C)x + A + B + D \quad = \quad 1.$$

The solution to the resulting system is

$$A = \frac{1}{4}, \quad B = \frac{1}{4}, \quad C = 0, \quad D = \frac{1}{2}.$$

We get

$$\begin{aligned}
\int \frac{1}{1-x^4}\,dx \quad &= \quad \int \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}\,dx \\
&= \quad -\frac{1}{4}\ln|1-x| + \frac{1}{4}\ln|1+x| + \frac{1}{2}\tan^{-1}x + C \\
&= \quad \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\tan^{-1}x + C.
\end{aligned}$$

Problem: Compute $\displaystyle\int \frac{4x+3}{(x^2-2x+5)^{\frac{3}{2}}}\,dx$.

Solution: We complete the square in the denominator to get

$$x^2 - 2x + 5 \quad = \quad (x-1)^2 + 4.$$

We then use the substitution $u = x - 1$, $du = dx$ to rewrite the integral. We will need to know that $x = u + 1$. We get

$$\begin{aligned}
\int \frac{4x+3}{(x^2-2x+5)^{\frac{3}{2}}}\,dx \quad &= \quad \int \frac{4(u+1)+3}{(u^2+4)^{\frac{3}{2}}}, du \\
&= \quad \int \frac{4u+7}{(u^2+4)^{\frac{3}{2}}}\,du.
\end{aligned}$$

We split this up into two integrals, getting

$$\int \frac{4u+7}{(u^2+4)^{\frac{3}{2}}}\,du \quad = \quad \int \frac{4u}{(u^2+4)^{\frac{3}{2}}}\,du + \int \frac{7}{(u^2+4)^{\frac{3}{2}}}\,du. \tag{1}$$

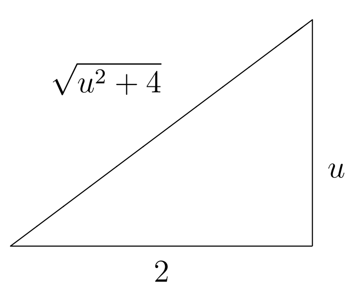
The first of these integrals will yield to the substitution $w = u^2 + 4$, $dw = 2u\,du$. We get

$$\begin{aligned}
\int \frac{4u}{(u^2+4)^{\frac{3}{2}}}\,du &= 2\int \frac{dw}{w^{\frac{3}{2}}} \\
&= -\frac{4}{\sqrt{w}} \\
&= -\frac{4}{\sqrt{u^2+4}}.
\end{aligned}$$

The second integral in line (1) requires a trigonometric substitution. We let $u = 2\tan\theta$, so that $du = 2\sec^2\theta\,d\theta$. Making this substitution, we get

$$\begin{aligned}
\int \frac{7}{(u^2+4)^{\frac{3}{2}}}\,du &= \int \frac{14\sec^2\theta}{(4\sec^2\theta)^{\frac{3}{2}}}\,d\theta \\
&= \frac{7}{4}\int \frac{1}{\sec\theta}\,d\theta \\
&= \frac{7}{4}\int \cos\theta\,d\theta \\
&= \frac{7}{4}\sin\theta.
\end{aligned}$$

Recalling that $\tan\theta = \dfrac{u}{2}$, we make a sketch:



From the picture, we find that $\sin\theta = \dfrac{u}{\sqrt{u^2+4}}$. Thus the second integral in line (1) is

$$\int \frac{7}{(u^2+4)^{\frac{3}{2}}}\,du = \frac{7}{4}\sin\theta = \frac{7u}{4\sqrt{u^2+4}}.$$

Putting the pieces together and undoing the u-subsitution, we get

$$\begin{aligned}
\int \frac{4x+3}{(x^2-2x+5)^{\frac{3}{2}}}\,dx &= -\frac{4}{\sqrt{u^2+4}} + \frac{7u}{4\sqrt{u^2+4}} + C \\
&= -\frac{16}{4\sqrt{x^2-2x+5}} + \frac{7x-7}{4\sqrt{x^2-2x+5}} + C \\
&= \frac{7x-23}{4\sqrt{x^2-2x+5}} + C.
\end{aligned}$$

Problem: Compute $\displaystyle\int \frac{3x^3+x^2+28x+11}{x^2+9}\,dx$.

Solution: Since the degree of the numerator is bigger than the degree of the denominator, we begin by dividing x^2+9 into $3x^3+x^2+28x+11$. We get

$$\begin{array}{rrrrrrrrr}
 & & & 3x & + & 1 & & & \\ \cline{4-9}
x^2 & + & 9 \big) & 3x^3 & + & x^2 & + & 28x & + \; 11 \\
 & & & 3x^3 & & & + & 27x & \\ \cline{4-8}
 & & & & & x^2 & + & x & + \; 11 \\
 & & & & & x^2 & & & + \; 9 \\ \cline{6-9}
 & & & & & & & x & + \; 2
\end{array}$$

Thus

$$\begin{aligned}
\int \frac{3x^3+x^2+28x+11}{x^2+9}\,dx &= \int 3x+1+\frac{x+2}{x^2+9}\,dx \qquad (2)\\
&= \int 3x+1\,dx + \int \frac{x}{x^2+9}\,dx + \int \frac{2}{x^2+9}\,dx \qquad (3)
\end{aligned}$$

For the second integral in line (3), we use the substitution $u = x^2+9$, $du = 2x\,dx$. We get

$$\begin{aligned}
\int \frac{x}{x^2+9}\,dx &= \frac{1}{2}\int \frac{du}{u} \\
&= \frac{1}{2}\ln|u| \\
&= \frac{1}{2}\ln(x^2+9).
\end{aligned}$$

For the third integral in line (3), we use the trig substitution $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$. We get

$$\begin{aligned}
\int \frac{2}{x^2+9}\, dx &= \int \frac{6\sec^2\theta}{9\sec^2\theta}\, d\theta \\
&= \frac{2}{3}\int d\theta \\
&= \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right).
\end{aligned}$$

Integrating the first expression in line (3) and putting everything together, we get

$$\int \frac{3x^3 + x^2 + 28x + 11}{x^2+9}\, dx = \frac{3x^2}{2} + x + \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right).$$