Math 202(01) Practice Hour Exam #1 – Solution 18 February 2002

1. If $\sinh x = \dfrac{5}{12}$, find $\cosh x$ and $\tanh x$.

   Solution: Using the identity $\cosh^2 x - \sinh^2 x = 1$, we find that

$$\cosh^2 x - \frac{25}{144} = 1$$

   so that $\cosh^2 x = \dfrac{169}{144}$, so that $\cosh x = \dfrac{13}{12}$. Then we get

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{5}{13}. \end{aligned}$$

2. Find $\lim\limits_{x\to\infty} (e^x + x)^{\frac{1}{x}}$.

   Solution: The limit has the indeterminate form ∞^0. We set

$$y = \lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$$

   and take logarithms to get

$$\ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x}.$$

   The limit on the right has the form ∞/∞, so we apply l'Hospital's rule to get

$$\ln y = \lim_{x\to\infty} \frac{e^x + 1}{e^x + x}.$$

   The limit still has the form ∞/∞, so we again apply l'Hospital's rule to get

$$\ln y = \lim_{x\to\infty} \frac{e^x}{e^x + 1}.$$

   Finally, we can multiply top and bottom by e^{-x} to get

$$\begin{aligned} \ln y &= \lim_{x\to\infty} \frac{1}{1 + e^{-x}} \\ &= 1 \end{aligned}$$

   Thus $y = e$.

3. Compute $\int_1^5 \sqrt{3x+1}\,dx$.

   Solution: Let $u = 3x+1$ so that $du = 3\,dx$. We have

$$\begin{aligned}\int \sqrt{3x+1}\,dx &= \frac{1}{3}\int \sqrt{u}\,du \\ &= \frac{2u^{\frac{3}{2}}}{9} + C \\ &= \frac{2}{9}(3x+1)^{\frac{3}{2}} + C\end{aligned}$$

   So that

$$\begin{aligned}\int_1^5 \sqrt{3x+1}\,dx &= \left[\frac{2}{9}(3x+1)^{\frac{3}{2}}\right]_1^5 \\ &= \frac{2}{9}\left(16^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) \\ &= \frac{2}{9}(56) \\ &= \frac{112}{9}.\end{aligned}$$

4. Find $\int \frac{x^7}{\sqrt{x^4+6}}\,dx$.

   Solution: Let $u = x^4+6$ so that $du = 4x^3\,dx$. We will also need to know that $x^4 = u-6$. We get

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^4+6}}x^4\,dx &= \frac{1}{4}\int \frac{1}{\sqrt{u}}(u-6)\,du \\ &= \frac{1}{4}\int u^{\frac{1}{2}} - 6u^{-\frac{1}{2}}\,du \\ &= \frac{1}{4}\left(\frac{2u^{\frac{3}{2}}}{3} - 12u^{\frac{1}{2}}\right) \\ &= \frac{(x^4+6)^{\frac{3}{2}}}{6} - 3\sqrt{x^4+6} + C.\end{aligned}$$

5. Find $\int x^2 e^{2x}\,dx$.

Solution: We use integration by parts. Set

$$\begin{aligned} u &= x^2 & v &= \frac{e^{2x}}{2} \\ du &= 2x\,dx & dv &= e^{2x}\,dx. \end{aligned}$$

We get

$$\int x^2 e^{2x}\,dx = \frac{x^2 e^{2x}}{2} - \int x e^{2x}\,dx.$$

So we need to evaluate $\int x e^{2x}\,dx$. We do this by parts as well. Set

$$\begin{aligned} u &= x & v &= \frac{e^{2x}}{2} \\ du &= dx & dv &= e^{2x}\,dx. \end{aligned}$$

We get

$$\begin{aligned} \int x e^{2x}\,dx &= \frac{x e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx \\ &= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C. \end{aligned}$$

Substiuting this into the original integral, we get

$$\begin{aligned} \int x^2 e^{2x}\,dx &= \frac{x^2 e^{2x}}{2} - \left(\frac{x e^{2x}}{2} - \frac{e^{2x}}{4}\right) + C \\ &= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} + C. \end{aligned}$$

6. Find $\int \sin^{-1} x\,dx$.

Solution: We use integration by parts. Set

$$\begin{aligned} u &= \sin^{-1} x & v &= x \\ du &= \frac{dx}{\sqrt{1-x^2}} & dv &= dx. \end{aligned}$$

We get

$$\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

To evaluate the integral on the right, set $w = 1 - x^2$ so that $dw = -2x\,dx$. We get

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int w^{-\frac{1}{2}}\,dw \\
&= -\sqrt{w} + C \\
&= -\sqrt{1-x^2} + C.
\end{aligned}$$

We substitute this back into the original integral to get

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

7. Compute $\int \cos^3 x \sin^3 x\,dx$.

   Solution: We write

$$\int \cos^2 x \sin^3 x \cos x\,dx = \int (1-\sin^2 x)\sin^3 x \cos x\,dx.$$

   Set $u = \sin x$ so that $du = \cos x\,dx$. We get

$$\begin{aligned}
\int \cos^3 x \sin^3 x\,dx &= \int (1-u^2)u^3\,du \\
&= \int u^3 - u^5\,du \\
&= \frac{u^4}{4} - \frac{u^6}{6} + C \\
&= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C.
\end{aligned}$$

8. Compute $\int \tan^3 x\,dx$.

   Solution: We write

$$\begin{aligned}
\int \tan^3 x\,dx &= \int \tan^2 x \tan x\,dx \\
&= \int (\sec^2 x - 1)\tan x\,dx \\
&= \int \tan x \sec^2 x\,dx - \int \tan x\,dx \\
&= \frac{\tan^2 x}{2} - \ln|\sec x| + C.
\end{aligned}$$

9. Use a trig substitution to eliminate the square root from the integral $\int \frac{x^2}{\sqrt{4x - x^2}}\, dx$. Do not try to evaluate the integral; just rewrite it as an integral $d\theta$ with no square root.

Solution: To begin, we complete the square, writing

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x + 4 - 4) \\ &= -((x-2)^2 - 4) \\ &= 4 - (x-2)^2. \end{aligned}$$

We want $(x-2)^2 = 4\sin^2\theta$, so we'll set $x - 2 = 2\sin\theta$. That is,

$$\begin{aligned} x &= 2 + 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}$$

The subsitution yields

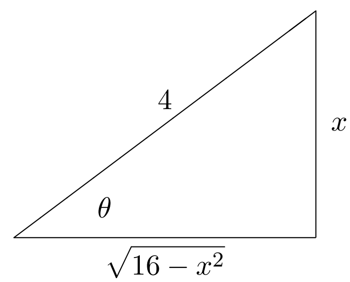
$$\begin{aligned} \int \frac{x^2}{\sqrt{4x - x^2}}\, dx &= \int \frac{(2 + 2\sin\theta)^2 2\cos\theta\, d\theta}{\sqrt{4 - 4\sin^2\theta}} \\ &= \int \frac{(2 + 2\sin\theta)^2 2\cos\theta}{2\cos\theta}\, d\theta \\ &= \int (2 + 2\sin\theta)^2\, d\theta. \end{aligned}$$

10. Compute $\int \frac{x^3}{\sqrt{16 - x^2}}\, dx$. (Use a trig substitution.)

Solution: We'll use the trig substitution $x = 4\sin\theta$, so that $dx = 4\cos\theta\, d\theta$. Using the substitution, we get

$$\begin{aligned} \int \frac{x^3}{\sqrt{16 - x^2}}\, dx &= \int \frac{64\sin^3\theta}{\sqrt{16 - 16\sin^2\theta}} 4\cos\theta\, d\theta \\ &= 64 \int \sin^3\theta\, d\theta \\ &= 64 \int (1 - \cos^2\theta)\sin\theta\, d\theta \\ &= -64\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C \\ &= \frac{64\cos^3}{3} - 64\cos\theta + C. \end{aligned}$$

To write $\cos\theta$ in terms of x, we draw a triangle containing an angle θ with $x = 4\sin\theta$.



We have $\cos\theta = \dfrac{\sqrt{16-x^2}}{4}$, so that

$$\frac{64\cos^3}{3} - 64\cos\theta + C \quad = \quad \frac{(16-x^2)^{\frac{3}{2}}}{3} - 16\sqrt{16-x^2} + C.$$