1. Compute $\displaystyle\int \frac{6x^2+23x+16}{(x+2)^2(x-1)}\,dx$.

   Solution: We use partial fractions. We have

$$\frac{6x^2+23x+16}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1}.$$

   Multiplying through to clear denominators, we get

$$\begin{aligned} 6x^2+23x+16 &= A(x+2)(x-1) + B(x-1) + C(x+2)^2 \\ &= A(x^2+x-2) + B(x-1) + C(x^2+4x+4) \\ &= (A+C)x^2 + (A+B+4C)x + (-2A-B+4C). \end{aligned}$$

   We get the system

$$\begin{array}{rcrcrcr} A & & & + & C & = & 6 \\ A & + & B & + & 4C & = & 23 \\ -2A & - & B & + & 4C & = & 16 \end{array}$$

   The calculator says the solution is $A = 1$, $B = 2$, and $C = 5$. We get

$$\begin{aligned} \int \frac{6x^2+23x+16}{(x+2)^2(x-1)}\,dx &= \int \frac{1}{x+2} + \frac{2}{(x+2)^2} + \frac{5}{x-1}\,dx \\ &= \ln|x+2| - \frac{2}{x+2} + 5\ln|x-1| + C. \end{aligned}$$

2. Compute $\displaystyle\int \frac{x^5+5x^3+5x+4}{x^2+1}\,dx$.

   Solution: The degree of the numerator is greater than the degree of the denominator, so we begin by dividing. We have

$$\begin{array}{r|l} & x^3 + 4x - \dfrac{x+4}{x^2+1} \\ x^2+1 & x^5 + 5x^3 + 5x + 4 \\ & \underline{x^5 + x^3} \\ & \quad 4x^3 + 5x \\ & \quad \underline{4x^3 + 4x} \\ & \qquad\quad x + 4 \end{array}$$

Thus

$$\begin{aligned} \int \frac{x^5+5x^3+5x+4}{x^2+1}\,dx &= \int x^3+4x+\frac{x}{x^2+1}+\frac{4}{x^2+1}\,dx \\ &= \frac{x^4}{4}+2x^2+\frac{1}{2}\ln(x^2+1)+4\tan^{-1}(x)+C. \end{aligned}$$

3. Use the midpoint rule with 50 sub-intervals to estimate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$. (Note that the upper limit of integration is $\frac{1}{2}$, not 1.) Round your answer to six decimal places.

Solution: Let $f(x) = \frac{1}{\sqrt{1-x^2}}$. We have

$$M_{50} = \frac{1}{100}\left[f\left(\frac{1}{200}\right)+f\left(\frac{3}{200}\right)+f\left(\frac{5}{200}\right)+\cdots+f\left(\frac{99}{200}\right)\right].$$

This is easy to estimate on the calculator; we get

$$M_{50} \approx 0.523596.$$

4. Consider $\int_0^2 \tan^{-1} x\,dx$.

(a) Estimate the value of the integral using trapezoids with $n = 10$. Round your answer to six decimal places.
Solution: Let $f(x) = \tan^{-1} x$. We have

$$T_{10} = \frac{2}{20}\left[f(0)+2f\left(\frac{1}{5}\right)+2f\left(\frac{2}{5}\right)+\cdots+2f\left(\frac{9}{5}\right)+f(2)\right].$$

Estimating this on the calculator, we get

$$T_{10} \approx 1.406907.$$

(b) Find an upper bound for the absolute value of the error in your approximation.

Solution: We'll need K_2. We have

$$\begin{aligned} f(x) &= \tan^{-1} x \\ f'(x) &= \frac{1}{x^2+1} \\ f''(x) &= -\frac{2x}{(x^2+1)^2} \end{aligned}$$

so that $|f''(x)| = \dfrac{2x}{(x^2+1)^2} = (2x) \cdot \dfrac{1}{(x^2+1)^2}$. On the interval $0 \le x \le 2$, the first factor is always less than or equal to 4, and the second is always less than or equal to 1. So we may take $K_2 = 4$. Our error bound is

$$\begin{aligned} E_T &\le \frac{4(2-0)^3}{12 \times 10^2} \\ &= \frac{4 \times 8}{12 \times 10^2} \\ &= \frac{8}{300} \\ &\approx 0.0266667. \end{aligned}$$

We could find a better value for K_2 by computing

$$f'''(x) = \frac{2-6x^2}{(1+x^2)^3}$$

and noting that the function $f''(x)$ on $[0,2]$ has critical numbers at 0, $\sqrt{\frac{1}{3}}$ and 2. We have

$$\begin{aligned} f''(0) &= 0 \\ f''\left(\sqrt{\frac{1}{3}}\right) &= -\frac{3\sqrt{3}}{8} \\ &\approx -0.649519 \\ f''(2) &= -\frac{4}{25} \\ &= -0.16. \end{aligned}$$

Thus the maximum value of $|f''(x)|$ on $[0,2]$ is $\dfrac{3\sqrt{3}}{8}$, and we may take $K_2 = \dfrac{3\sqrt{3}}{8}$. In this case, our error estimate becomes

$$E_T \le \frac{\frac{3\sqrt{3}}{8} \times 8}{12 \times 10^2}$$

$$
\begin{aligned}
&= \frac{3\sqrt{3}}{1200} \\
&\approx 0.00433013.
\end{aligned}
$$

(c) What is the smallest value of n we may use in a trapezoid approximation to guarantee an error of less than 10^{-4}?

Solution: We need to solve the inequality

$$\frac{K_2 \times 8}{12n^2} \quad < \quad 10^{-4}$$

which simplifies to

$$\frac{2K_2}{3n^2} \quad < \quad 10^{-4},$$

or

$$n \quad > \quad \sqrt{\frac{2 \times 10^4 \times K_2}{3}}.$$

With $K_2 = 4$, the number on the right is approximately 163.3. We may take $n = 164$ to achieve the desired accuracy. If we have done the extra work to get $K_2 = \frac{3\sqrt{3}}{8}$, then the number on the right evaluates to about 65.8, so we know that $n = 66$ sub-intervals will give the desired accuracy.

5. Rewrite the integral $\int_3^\infty xe^{-x}\,dx$ as a limit, and then find its value.

Solution: The integral is improper, and we get

$$\int_3^\infty xe^{-x}\,dx \quad = \quad \lim_{t\to\infty} \int_3^t xe^{-x}\,dx.$$

To find the value, we apply integration by parts. With

$$
\begin{aligned}
u &= x & v &= -e^{-x} \\
du &= dx & dv &= e^{-x},
\end{aligned}
$$

we get

$$
\begin{aligned}
\int xe^{-x}\,dx &= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x}.
\end{aligned}
$$

Thus

$$\begin{aligned}
\int_3^\infty xe^{-x}\,dx &= \lim_{t\to\infty}\left[-te^{-t} - e^{-t} + 3e^{-3} + e^{-3}\right] \\
&= 3e^{-3} + e^{-3} - \lim_{t\to\infty} e^{-t} - \lim_{t\to\infty} te^{-t}.
\end{aligned}$$

The first limit is clearly zero, and the second can be written as

$$\lim_{t\to\infty} \frac{t}{e^t},$$

which has the form $\frac{\infty}{\infty}$. We apply l'Hospital's rule to get

$$\lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t},$$

and the right-hand limit is clearly zero. Thus

$$\int_3^\infty xe^{-x}\,dx = 4e^{-3}.$$

6. Evaluate the definite integral $\displaystyle\int_1^3 \frac{1}{x^2-2x}\,dx$.

Solution: We notice that the integrand has a vertical asymptote as $x = 2$, and since this lies in the interval over which we are integrating, we will need to treat this as an improper integral. We have

$$\begin{aligned}
\int_1^3 \frac{1}{x^2-2x}\,dx &= \int_1^2 \frac{1}{x^2-2x}\,dx + \int_2^3 \frac{1}{x^2-2x}\,dx \\
&= \lim_{t\to 2^-} \int_1^t \frac{1}{x^2-2x}\,dx + \lim_{t\to 2^+} \int_t^3 \frac{1}{x^2-2x}\,dx
\end{aligned}$$

To carry out either integration, we will need to find the indefinite integral of $\frac{1}{x^2-2x}$. We do this using partial fractions. We have

$$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

so that

$$1 = A(x-2) + Bx.$$

From this, we get $A = -\dfrac{1}{2}$ and $B = \dfrac{1}{2}$. Thus

$$\begin{aligned}
\int \frac{dx}{x^2 - 2x} &= \frac{1}{2} \int \frac{1}{x-2} - \frac{1}{x}\, dx \\
&= \frac{1}{2} \left(\ln|x-2| - \ln|x| \right) + C \\
&= \frac{1}{2} \ln\left| \frac{x-2}{x} \right| + C.
\end{aligned}$$

The first of the definite integrals above becomes

$$\begin{aligned}
\lim_{t \to 2^-} \int_1^t \frac{1}{x^2 - 2x}\, dx &= \frac{1}{2} \lim_{t \to 2^-} \left[\ln\left| \frac{t-2}{t} \right| - \ln\left| \frac{1-2}{1} \right| \right] \\
&= \frac{1}{2} \lim_{t \to 2^-} \left[\ln\left| 1 - \frac{2}{t} \right| - \ln(1) \right] \\
&= \frac{1}{2} \lim_{t \to 2^-} \ln\left(1 - \frac{2}{t} \right).
\end{aligned}$$

The limit has the form $\ln(0)$, which indicates that it is an infinite limit. The first part of the improper integral diverges, so we must conclude that the entire integral diverges.