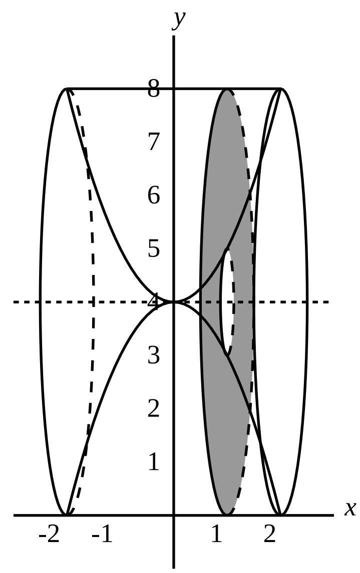
1. (a) Let R be the region bounded by the curve $y = 4 - x^2$ and the x-axis. Set up, but do not evaluate, an integral for the volume generated when R is revolved about the line $y = 4$.

Solution: We use the washer method. The washer at position x has

thickness dx
inner radius $4 - (4 - x^2) = x^2$
outer radius $4 - 0 = 4$

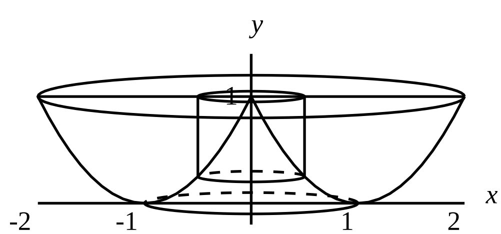
The volume is given by

$$\begin{aligned} V &= \int_{-2}^{2} \pi(4^2 - (x^2)^2)\,dx \\ &= \pi \int_{-2}^{2} 16 - x^4\,dx \end{aligned}$$



(b) Let R be the region bounded by the parabola $y = (x-1)^2$ and the line $y = 1$. Set up, but do not evaluate, an integral for the volume generated when R is revolved about the y-axis.

Solution: The given parabola and line intersect in two points, $(0, 1)$ and $(2, 1)$.



We set up the integral using the shells method. The shell at position x has

thickness dx

radius x
height $1-(x-1)^2$.

The volume is given by

$$V = \int_0^2 2\pi x(1-(x-1)^2)\,dx$$

2. Set up, but do not evaluate, an integral for the area of the surface generated when the part of the curve $y=(\ln x)^2$ with $1 \le x \le 4$ is revolved about the x-axis.

Solution: We have $y'(x) = \dfrac{2\ln x}{x}$, so that $(y'(x))^2 = \dfrac{4(\ln x)^2}{x^2}$. The band at position x has radius $(\ln x)^2$ and slant height

$$\sqrt{1+\frac{4(\ln x)^2}{x^2}}\,dx.$$

The surface area is given by

$$S = 2\pi\int_1^4 (\ln x)^2\sqrt{1+\frac{4(\ln x)^2}{x^2}}\,dx.$$

3. (a) Consider the sequence $\{a_n\}$ given by $a_n = 3^n 4^{1-n}$.
Find $\lim_{n\to\infty} a_n$.

Solution: We have $a_n = 4\times\left(\dfrac{3}{4}\right)^n$. Since $\lim_{n\to\infty} r^n = 0$ if $|r|<1$, we have

$$\begin{aligned}\lim_{n\to\infty} a_n &= 4\lim_{a\to\infty}\left(\frac{3}{4}\right)^n \\ &= 4\times 0 = 0.\end{aligned}$$

(b) Consider the sequence $\{a_n\}$ given by $a_n = \dfrac{\sin n}{n^2}$
Find $\lim_{n\to\infty} a_n$.

Solution: Since $|\sin n| \le 1$ for all n, we know that $|a_n| < \dfrac{1}{n^2}$ for every n. This implies that

$$\lim_{n\to\infty}|a_n| \le \lim_{n\to\infty}\frac{1}{n^2} = 0,$$

so that $\lim_{n\to\infty}|a_n| = 0$. Thus $\lim_{n\to\infty} a_n = 0$.

4. (a) Find the sum of the series

$$\frac{3}{4} - \frac{6}{12} + \frac{12}{36} - \frac{24}{108} + \cdots,$$

assuming the pattern continues.

Solution: This is a geometric series with $a = \dfrac{3}{4}$ and $r = -\dfrac{2}{3}$. So the sum is

$$\begin{aligned} \frac{a}{1-r} &= \frac{3}{4} \times \frac{1}{1+\frac{2}{3}} \\ &= \frac{3}{4} \times \frac{3}{5} \\ &= \frac{9}{20}. \end{aligned}$$

(b) Let $x = 3.513513513513\ldots$. Write x as a quotient of two integers.

Solution: We can write x as a geometric series:

$$x = 351 \times \frac{1}{10^2} + 351 \times \frac{1}{10^5} + 351 \times \frac{1}{10^8} + \cdots.$$

We have $a = \dfrac{351}{100}$ and $r = \dfrac{1}{1000}$, so the sum of the series is

$$\begin{aligned} \frac{a}{1-r} &= \frac{351}{100} \times \frac{1}{1 - \frac{1}{1000}} \\ &= \frac{351}{100} \times \frac{1000}{999} \\ &= \frac{3510}{999} \\ &= \frac{130}{37}. \end{aligned}$$

5. Determine whether each series converges or diverges. Give reasons.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3n+5}$

Solution: Since

$$\lim_{n\to\infty} \frac{n}{3n+5} = \frac{1}{3} \neq 0$$

this series diverges by the Divergence Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

Solution: Seeing no other obvious test to use, we consider applying the integral test. The function $f(x) = \dfrac{x^2}{e^x}$ is clearly positive for $x \geq 1$. To determine whether it is decreasing, we take a derivative, getting

$$\begin{aligned} f'(x) &= \frac{2xe^x - x^2e^x}{e^{2x}} \\ &= \frac{x(2-x)}{e^x}. \end{aligned}$$

If $x > 2$, then $f'(x)$ is negative. This means that the sequence of terms $\dfrac{n^2}{e^n}$ is decreasing for $n \geq 2$, so we may apply the integral test.
We compute

$$\int_1^{\infty} \frac{x^2}{e^x}\,dx.$$

Let's attack the indefinite integral $\displaystyle\int x^2e^{-x}\,dx$ first. We use integration by parts, letting $u = x^2$ and $dx = e^{-x}\,dv$. Then $du = 2x\,dx$ and $v = -e^{-x}$. Thus

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx.$$

We use integration by parts on the rightmost integral, letting $u = x$ and $dv = e^{-x}\,dx$, so that $du = dx$ and $v = -e^{-x}$. We get

$$\begin{aligned} \int x^2e^{-x}\,dx &= -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right] \\ &= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C. \end{aligned}$$

Thus

$$\begin{aligned} \int_1^{\infty} x^2e^{-x}\,dx &= \lim_{t\to\infty}\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t} + e^{-1} + 2e^{-1} + 2e^{-1}\right] \\ &= 5e^{-1} - \lim_{t\to\infty}\frac{t^2}{e^t} - 2\lim_{t\to\infty}\frac{t}{e^t} - 2\lim_{t\to\infty}\frac{1}{e^t}. \end{aligned}$$

The third limit is clearly zero; each of the first two has the form ∞/∞, so we may apply l'Hospital's rule. We get

$$\lim_{t\to\infty}\frac{t}{e^t} = \lim_{t\to\infty}\frac{1}{e^t} = 0$$

and

$$\lim_{t\to\infty}\frac{t^2}{e^t}=\lim_{t\to\infty}\frac{2t}{e^t}=\lim_{t\to\infty}\frac{2}{e^t}=0.$$

Thus the integral converges (its value is $5e^{-1}$), and so the series is convergent, as well.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n+2}$

Solution: We might try comparing this series with the geometric series $\sum\frac{1}{3^n}$.
For $n\geq 0$, we know that $3^n+2>3^n$, so that

$$\frac{1}{3^n+2}<\frac{1}{3^n}.$$

We also know that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{3^n}$ is convergent (it's a geometric series with $|r|<1$), so we conclude that the given series is convergent by the Basic Comparison Test.

(d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n-1}}$

Solution: We'll try limit comparison with the p-series $\sum\frac{1}{n^{\frac{3}{2}}}$.
Let $a_n=\dfrac{1}{n\sqrt{n-1}}$ and let $b_n=\dfrac{1}{n^{\frac{3}{2}}}$. Then

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\frac{1}{n\sqrt{n-1}}\cdot\frac{n^{\frac{3}{2}}}{1}\\&=\lim_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{n^2(n-1)}}\\&=\lim_{n\to\infty}\sqrt{\frac{n^3}{n^3-n^2}}\\&=\sqrt{\lim_{n\to\infty}\frac{n^3}{n^3-n^2}}\\&=\sqrt{1}=1.\end{aligned}$$

Since 1 is finite and non-zero, we can use the Limit Comparison Theorem, along with the fact that $\sum\frac{1}{n^{\frac{3}{2}}}$ converges (it is a p-series with $p>1$) to conclude that the given series is convergent.

(e) $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{2^n}$

Solution: This is an alternating series, so we apply the alternating series test. The absolute values of the terms are given by

$$a_n = \frac{n}{2^n}$$

First we check that $\lim_{n\to\infty} a_n = 0$. Let $f(x) = \frac{x}{2^x}$. Then $\lim_{x\to\infty} f(x)$ has the form ∞/∞, so we may apply l'Hospital's rule. We get

$$\lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{2^x \ln 2}.$$

Since the second limit has the form $\frac{1}{\infty}$, we conclude that the function goes to zero as $x \to \infty$. Next we verify that the function (and thus the sequence a_n) is decreasing. We could use calculus for this, but here's an eaiser method. Consider

$$\begin{aligned} a_n - a_{n+1} &= \frac{n}{2^n} - \frac{n+1}{2^{n+1}} \\ &= \frac{2n}{2^{n+1}} - \frac{n+1}{2^{n+1}} \\ &= \frac{2n-n-1}{2^{n+1}} \\ &= \frac{n-1}{2^{n+1}}. \end{aligned}$$

For $n > 1$, this number is clearly positive. This shows that $a_n - a_{n+1} > 0$ for all $n > 1$, and thus that $a_n > a_{n+1}$ for all $n > 1$. That is, the sequence a_n is decreasing. Thus by the alternating series test, the series $\sum(-1)^{n-1}a_n$ is convergent.

6. Let $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$. (The series is convergent by the Alternating Series Test.) How many terms of the series are necessary to estimate s with an error of no more than 0.001? Use a calculator to estimate s with an error of no more than 0.001.

Solution: Since this series is alternating and the absolute values of the terms are decreasing, we know that for any partial sum s_n, we have

$$|s - s_n| \le b_{n+1}$$

where b_{n+1} is the absolute value of the $(n+1)^{\text{st}}$ terms of the series. Thus we need to solve $b_{n+1} < \frac{1}{1000}$, that is

$$\frac{1}{(n+1)^{\frac{3}{2}}} \le \frac{1}{1000}$$

$$\begin{aligned}(n+1)^{\frac{3}{2}} &\geq 1000 \\ n+1 &\geq 1000^{\frac{2}{3}} \\ &= 100.\end{aligned}$$

To achieve the desired accuracy, we need to add up the first 99 terms. We get

`sum seq((¯1)^(N-1)/N^(3/2),N,1,99,1)` $\approx$ 0.76565