1. Determine whether the given series is absolutely convergent, conditionally convergent, or divergent.

(a) \( \sum_{n=1}^{\infty} \frac{n^2}{(-2)^n} \).

Solution: We apply the ratio test. We have
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{2(n+1)^2} \cdot \frac{2^n}{n^2} = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2n+1} = \frac{1}{2} n \to \infty \frac{(n+1)^2}{n^2} = \frac{1}{2}.
\]

Since \( \frac{1}{2} < 1 \), we conclude that the series converges absolutely.

(b) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \).

Solution: Since both \( n \) and \( \ln n \) are increasing functions of \( n \), we know that \( \frac{1}{n \ln n} \) is a decreasing (and positive) function of \( n \). Furthermore, we know that
\[
\lim_{n \to \infty} \frac{1}{n \ln n} = 0
\]
because it has the form \( 1/\infty \). Thus by the Alternating Series Test, the given series converges. To test for absolute convergence, we consider the series
\[
\sum_{n=2}^{\infty} \frac{1}{n \ln n}.
\]
We let \( f(x) = \frac{1}{x \ln x} \), and apply the integral test. First, since \( x \) and \( \ln x \) are both increasing functions of \( x \), we know that \( \frac{1}{x \ln x} \) is decreasing (and positive), so that the
The integral test applies. We have
\[
\int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} [\ln(\ln x)]_0^t \\
= \lim_{t \to \infty} [\ln(\ln t) - \ln(\ln 2)].
\]
This limit is infinite. Since the integral diverges, we conclude that the series also diverges. In conclusion, the given series converges conditionally.

\[\sum_{n=0}^{\infty} \left( \frac{3 - n^2}{3n^2 + 2} \right)^n.\]
Solution: We apply the root test. We have
\[
\lim_{n \to \infty} \left[ \frac{3 - n^2}{3n^2 + 2} \right]^{\frac{1}{n}} = \lim_{n \to \infty} \left| \frac{3 - n^2}{3n^2 + 2} \right| \\
= \lim_{n \to \infty} \left| \frac{3/n^2 - 1}{3 + 2/n^2} \right| \\
= \left| \frac{-1}{3} \right| \\
= \frac{1}{3}.
\]
Since \(\frac{1}{3} < 1\), we conclude that the given series converges absolutely.

2. Find a power-series representation of the function \(x \ln(2 + x)\). Be sure to indicate the interval on which your power series is valid.
Solution: We have
\[
\frac{1}{2 + x} = \frac{1}{2} \cdot \frac{1}{1 + \frac{x}{2}} \\
= \frac{1}{2} \left( \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n \right) \\
= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}}
\]
for \(|x| < 2\). Next we integrate both sides of the equation above to get
\[
\ln(2 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n + 1)2^{n+1}} + C
\]
and evaluate both sides at \( x = 0 \) to find that

\[
\ln(2) = \mathcal{C}.
\]

Thus we have

\[
\ln(2 + x) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}} x^{n+1}.
\]

Finally, we multiply both sides by \( x \) to get

\[
x \ln(2 + x) = x \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}} x^{n+2}.
\]

The power series is valid for \( |x| < 2 \).

3. Determine the interval of convergence for the given power series.

(a) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1} \).

Solution: We apply the ratio test. We have

\[
\lim_{n \to \infty} \left| \frac{x^{2n+3}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n+1}} \right| = \lim_{n \to \infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0.
\]

This series converges for all values of \( x \); the interval of convergence is \((-\infty, \infty)\).

(b) \( \sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{3^n \sqrt{n}} \).

Solution: We apply the ratio test. We have

\[
\lim_{n \to \infty} \left| \frac{(x-2)^{2n+2}}{3^{n+1} \sqrt{n+1}} \cdot \frac{3^n \sqrt{n}}{x-2} \right| = \frac{|x-2|^2}{3} \lim_{n \to \infty} \sqrt{n} = \frac{|x-2|^2}{3}.
\]

The series converges absolutely for \( |x-2| < \sqrt{3} \), that is, for

\[2 - \sqrt{3} < x < 2 + \sqrt{3}.
\]

Checking the endpoints, we find that for \( x = 2 - \sqrt{3} \), the series becomes

\[
\sum_{n=1}^{\infty} \frac{(-\sqrt{3})^{2n}}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.
\]
which is divergent (it’s a $p$-series with $p < 1$). At $x = 2 + \sqrt{3}$, the series becomes

$$
\sum_{n=1}^{\infty} \frac{(\sqrt{3})^{2n}}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}
$$

which is still divergent. The interval of convergence is $2 - \sqrt{3} < x < 2 + \sqrt{3}$.

4. Use a power series to estimate the value of $\int_{0}^{\frac{1}{2}} \frac{x}{1 + x^4} \, dx$ with an error of less than $10^{-6}$.

Solution: We have

$$
\frac{1}{1 + x^4} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}
$$

so that

$$
\frac{x}{1 + x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n+1}.
$$

The series expansion is valid for $|x| < 1$. Thus

$$
\int_{0}^{\frac{1}{2}} \frac{x}{1 + x^4} \, dx = \left[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{4n + 2} \right]_{0}^{\frac{1}{2}}
$$

$$
= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n + 2) \cdot 2^{4n+2}}
$$

$$
= \sum_{n=0}^{\infty} \frac{1}{2 \cdot 2^2} - \frac{1}{6 \cdot 2^6} + \frac{1}{10 \cdot 2^{10}} - \cdots
$$

The first term in this sequence whose absolute value is less than $10^{-6}$ is $\frac{1}{18 \cdot 2^{18}}$. We approximate the integral as

$$
\int_{0}^{\frac{1}{2}} \frac{x}{1 + x^4} \, dx \approx \frac{1}{2 \cdot 2^2} - \frac{1}{6 \cdot 6^6} + \frac{1}{10 \cdot 2^{10}} - \frac{1}{14 \cdot 2^{14}}
$$

$$
\approx 0.122489.
$$
5. (a) Find the third-order Taylor polynomial for \( f(x) = \tan x \) about \( \pi/4 \).

Solution: We have

\[
\begin{align*}
  f(x) &= \tan x \\
  f'(x) &= \sec^2 x \\
  f''(x) &= 2 \sec^2 x \tan x \\
  f'''(x) &= 2(\sec^4 x + 2 \sec^2 x \tan^2 x)
\end{align*}
\]

We’ll also need to know that \( \tan(\pi/4) = 1 \) and \( \sec(\pi/4) = \sqrt{2} \). We get

\[
\begin{align*}
  f(\pi/4) &= 1 \\
  f'(\pi/4) &= 2 \\
  f''(\pi/4) &= 4 \\
  f'''(\pi/4) &= 16
\end{align*}
\]

Thus the third-order Taylor polynomial, \( T_3(x) \) is given by

\[
T_3(x) = 1 + \frac{2}{1!} (x - \frac{\pi}{4}) + \frac{4}{2!} \left( x - \frac{\pi}{4} \right)^2 + \frac{16}{3!} \left( x - \frac{\pi}{4} \right)^3
\]

(b) Find the Taylor series for \( f(x) = \sqrt{x+1} \) at 0. Write the first three terms and the general form of the \( n \)th term.

Solution: We have

\[
\begin{align*}
  f(x) &= (x + 1)^{\frac{1}{2}} \\
  f'(x) &= \frac{1}{2} (x + 1)^{-\frac{1}{2}} \\
  f''(x) &= -\frac{1}{4} (x + 1)^{-\frac{3}{2}} \\
  f'''(x) &= \frac{3}{8} (x + 1)^{-\frac{5}{2}} \\
  f^{(4)}(x) &= -\frac{3 \cdot 5}{16} (x + 1)^{-\frac{7}{2}} \\
  f^{(5)}(x) &= \frac{3 \cdot 5 \cdot 7}{32} (x + 1)^{-\frac{9}{2}}
\end{align*}
\]

and so on. It appears that the \( n \)th derivative of \( f \) has the form

\[
f^{(n)}(x) = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} (x + 1)^{-\frac{2n-1}{2}}
\]
(at least for \( n \geq 2 \)). Evaluating all these derivatives at \( x = 0 \), we get

\[
\begin{align*}
f(0) &= 1 \\
f'(0) &= \frac{1}{2} \\
f''(0) &= -\frac{1}{4} \\
f'''(0) &= \frac{3}{8} \\
&\quad \vdots \\
f^{(n)}(0) &= (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}.
\end{align*}
\]

The Taylor series \( T(x) \) begins

\[
1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2 + \frac{3}{8 \cdot 3!}x^3 - \cdots
\]

and the general term is

\[
(-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!2^n} x^n.
\]