Practice Problems with Solutions **6 May 2002**

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1. Determine whether the given series is absolutely convergent, conditionally convergent, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(-2)^n}$.

   Solution: We apply the ratio test. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \\
&= \frac{1}{2}.
\end{aligned}$$

   Since $\frac{1}{2} < 1$, we conclude that the series converges absolutely.

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$.

   Solution: Since both n and $\ln n$ are increasing functions of n, we know that $\dfrac{1}{n \ln n}$ is a decreasing (and positive) function of n. Furthermore, we know that

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$$

   because it has the form $1/\infty$. Thus by the Alternating Series Test, the given series converges. To test for absolute convergence, we consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

   We let $f(x) = \dfrac{1}{x \ln x}$, and apply the integral test. First, since x and $\ln x$ are both increasing functions of x, we know that $\dfrac{1}{x \ln x}$ is decreasing (and positive), so that the

integral test applies. We have

$$\begin{aligned}
\int_2^\infty \frac{1}{x \ln x}\, dx &= \lim_{t\to\infty} \left[\ln(\ln x)\right]_0^t \\
&= \lim_{t\to\infty} \left[\ln(\ln t) - \ln(\ln 2)\right].
\end{aligned}$$

This limit is infinite. Since the integral diverges, we conclude that the series also diverges. In conclusion, the given series converges conditionally.

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3-n^2}{3n^2+2}\right)^n.$

Solution: We apply the root test. We have

$$\begin{aligned}
\lim_{n\to\infty} \left[\left|\frac{3-n^2}{3n^2+2}\right|^n\right]^{\frac{1}{n}} &= \lim_{n\to\infty} \left|\frac{3-n^2}{3n^2+2}\right| \\
&= \lim_{n\to\infty} \left|\frac{3/n^2-1}{3+2/n^2}\right| \\
&= \left|\frac{-1}{3}\right| \\
&= \frac{1}{3}.
\end{aligned}$$

Since $\frac{1}{3} < 1$, we conclude that the given series converges absolutely.

2. Find a power-series representation of the function $x\ln(2+x)$. Be sure to indicate the interval on which your power series is valid.

Solution: We have

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^{n+1}}
\end{aligned}$$

for $|x| < 2$. Next we integrate both sides of the equation above to get

$$\ln(2+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)2^{n+1}} + C$$

and evaluate both sides at $x = 0$ to find that

$$\ln(2) = C.$$

Thus we have

$$\ln(2+x) = \ln 2 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)2^{n+1}}.$$

Finally, we multiply both sides by x to get

$$x\ln(2+x) = x\ln 2 + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{(n+1)2^{n+1}}.$$

The power series is valid for $|x| < 2$.

3. Determine the interval of convergence for the given power series.

   (a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$.

   Solution: We apply the ratio test. We have

$$\begin{aligned} \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n+1}} &= \lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} \\ &= 0. \end{aligned}$$

   This series converges for all values of x; the interval of convergence is $(-\infty, \infty)$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{3^n\sqrt{n}}$.

   Solution: We apply the ratio test. We have

$$\begin{aligned} \lim_{n\to\infty} \frac{|x-2|^{2n+2}}{3^{n+1}\sqrt{n+1}} \cdot \frac{3^n\sqrt{n}}{|x-2|^{2n}} &= \frac{|x-2|^2}{3} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \\ &= \frac{|x-2|^2}{3}. \end{aligned}$$

   The series converges absolutely for $|x-2| < \sqrt{3}$, that is, for

$$2 - \sqrt{3} < x < 2 + \sqrt{3}.$$

   Checking the endpoints, we find that for $x = 2 - \sqrt{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-\sqrt{3})^{2n}}{3^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which is divergent (it's a p-series with $p < 1$). At $x = 2 + \sqrt{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(\sqrt{3})^{2n}}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which is still divergent. The interval of convergence is $2 - \sqrt{3} < x < 2 + \sqrt{3}$.

4. Use a power series to estimate the value of $\displaystyle\int_0^{\frac{1}{2}} \frac{x}{1+x^4}\, dx$ with an error of less than 10^{-6}.

   Solution: We have

$$\begin{aligned}\frac{1}{1+x^4} &= \sum_{n=0}^{\infty} (-x^4)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{4n}\end{aligned}$$

so that

$$\frac{x}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n+1}.$$

The series expansion is valid for $|x| < 1$. Thus

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{x}{1+x^4}\, dx &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{4n+2}\right]_0^{\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)\cdot 2^{4n+2}} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\cdot 2^2} - \frac{1}{6\cdot 2^6} + \frac{1}{10\cdot 2^{10}} - \cdots\end{aligned}$$

The first term in this sequence whose absolute value is less than 10^{-6} is $\dfrac{1}{18\cdot 2^{18}}$. We approximate the integral as

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{x}{1+x^4}\, dx &\approx \frac{1}{2\cdot 2^2} - \frac{1}{6\cdot 6^6} + \frac{1}{10\cdot 2^{10}} - \frac{1}{14\cdot 2^{14}} \\ &\approx 0.122489.\end{aligned}$$

5. (a) Find the third-order Taylor polynomial for $f(x) = \tan x$ about $\pi/4$.

Solution: We have

$$\begin{aligned} f(x) &= \tan x \\ f'(x) &= \sec^2 x \\ f''(x) &= 2\sec^2 x \tan x \\ f'''(x) &= 2(\sec^4 x + 2\sec^2 x \tan^2 x) \end{aligned}$$

We'll also need to know that $\tan(\pi/4) = 1$ and $\sec(\pi/4) = \sqrt{2}$. We get

$$\begin{aligned} f(\pi/4) &= 1 \\ f'(\pi/4) &= 2 \\ f''(\pi/4) &= 4 \\ f'''(\pi/4) &= 16 \end{aligned}$$

Thus the third-order Taylor polynomial, $T_3(x)$ is given by

$$\begin{aligned} T_3(x) &= 1 + \frac{2}{1!}\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3. \end{aligned}$$

(b) Find the Taylor series for $f(x) = \sqrt{x+1}$ at 0. Write the first three terms and the general form of the n^{th} term.

Solution: We have

$$\begin{aligned} f(x) &= (x+1)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(x+1)^{-\frac{1}{2}} \\ f''(x) &= -\frac{1}{4}(x+1)^{-\frac{3}{2}} \\ f'''(x) &= \frac{3}{8}(x+1)^{-\frac{5}{2}} \\ f^{(4)}(x) &= -\frac{3 \cdot 5}{16}(x+1)^{-\frac{7}{2}} \\ f^{(5)}(x) &= \frac{3 \cdot 5 \cdot 7}{32}(x+1)^{-\frac{9}{2}} \end{aligned}$$

and so on. It appears that the n^{th} derivative of f has the form

$$f^{(n)}(x) = (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n}(x+1)^{-\frac{2n-1}{2}}$$

(at least for $n \geq 2$). Evaluating all these derivatives at $x = 0$, we get

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= \frac{1}{2} \\
f''(0) &= -\frac{1}{4} \\
f'''(0) &= \frac{3}{8} \\
&\vdots \\
f^{(n)}(0) &= (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n}.
\end{aligned}$$

The Taylor series $T(x)$ begins

$$1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2 + \frac{3}{8 \cdot 3!}x^3 - \cdots$$

and the general term is

$$(-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{n!2^n}x^n.$$