Math 202(01) Solution
Quiz 9 3 May 2002

Show all work. An answer without sufficient work shown may not receive full credit even if it is correct.

1. Determine the interval of convergence for the power series $f(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 2^n}$.

   Solution: We apply the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{|x-3|^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{|x-3|^n}{n^2 2^n}} &= |x-3| \lim_{n\to\infty} \frac{n^2}{2(n+1)^2} \\ &= \frac{|x-3|}{2}.\end{aligned}$$

   This quantity is less than 1 when $|x-3| < 2$, that is, when $1 < x < 5$.

   Next we check the endpoints. At $x = 1$, we get the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

   which is convergent by the Alternating Series Test.

   At $x = 5$, we get the series

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

   which is convergent because it's a p-series with $p > 1$. The interval of convergence for the given series is $1 \le x \le 5$.

2. Starting with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, find a power series representation for the function

$$f(x) = \frac{x}{(1+2x)^2}.$$

   Be sure to indicate the interval on which your power series representation is valid.

Solution: We differentiate the geometric series formula to get

$$\frac{1}{(1-x)^2} \quad = \quad \sum_{n=0}^{\infty} n x^{n-1}$$

for $|x| < 1$. Next we substitute $-2x$ for x to get

$$\begin{aligned} \frac{1}{(1+2x)^2} \quad &= \quad \sum_{n=0}^{\infty} n(-2x)^{n-1} \\ &= \quad \sum_{n=0}^{\infty} (-1)^{n-1} n 2^{n-1} x^{n-1} \end{aligned}$$

for $|2x| < 1$, or $|x| < \frac{1}{2}$.

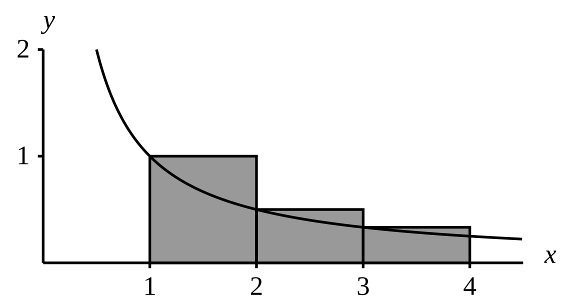
Finally, we multiply both sides by x to get

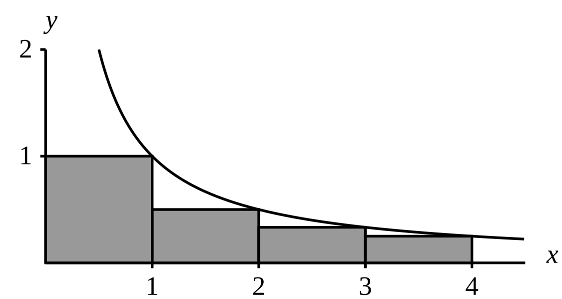
$$\frac{x}{(1+2x)^2} \quad = \quad \sum_{n=0}^{\infty} (-1)^{n-1} n 2^{n-1} x^{n}$$

for $|x| < \frac{1}{2}$.

Bonus problem: For approximately what value of N will $\displaystyle\sum_{k=1}^{N} \frac{1}{k}$ first exceed 20?

Solution 1:





The diagram on the left shows that

$$\sum_{k=1}^{N} \frac{1}{k} > \int_{1}^{N+1} \frac{dx}{x} = \ln(N+1).$$

The diagram on the right shows that

$$\sum_{k=1}^{N} \frac{1}{k} < 1 + \int_{1}^{N} \frac{dx}{x} = 1 + \ln(N).$$

From the first inequality, we know that the partial sum of the series will exceed 20 when $\ln(N+1) = 20$. That is, when $N = e^{20} - 1$. From the second, we know that the partial sum cannot reach 20 until $1 + \ln(N) = 20$. That is, until $\ln(N) = 19$, or $N = e^{19}$.

This says that the N^{th} partial sum of the series reaches 20 when N is somewhere between e^{19} and $e^{20} - 1$, or

$$178\,482\,300 < N < 485\,165\,196.$$

Any number in this range is a reasonable guess (and gets at least partial credit). We could take the average of these two numbers, but we know that the growth rate of the partial sums slows down as N increases, so the number half-way between 178 482 300 and 485 165 196 is probably too big.

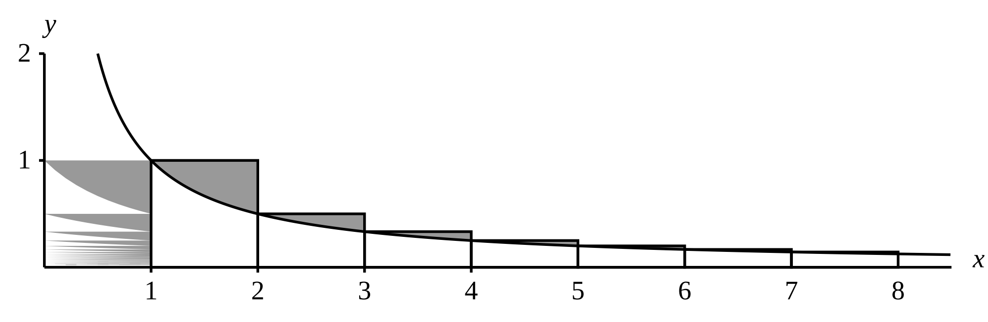
Since $\ln(N+1) \approx \ln(N)$ for large values of N, our reasoning above says that we are looking for a number N such that

$$19 < \ln(N) < 20.$$

Since the partial sums grow at approximately the same rate as $\ln(N)$, this time it makes sense to guess the middle number, that is, $e^{19.5} \approx 294\,267\,566$.

Solution 2: There's another way to see that $e^{19.5}$ is about the right number. We observe that $\sum_{k=1}^{N} \frac{1}{k} - \ln(N+1)$ is approximately $\frac{1}{2}$ (actually, a little more than $\frac{1}{2}$) for large values of N.

The shaded areas above the curve in the picture add up to $\sum_{k=1}^{N} \frac{1}{k} - \ln(N+1)$; all the wedges together fill up a little more than half of the one-by-one box along the y-axis.



Thus, for large values of N, we get

$$\sum_{k=1}^{N} \frac{1}{k} \approx \ln(N+1) + \frac{1}{2}.$$

The partial sums reach 20 when $\ln(N+1)$ is about 19.5. That is, when

$$N \approx e^{19.5} - 1 \approx 294\,267\,565.$$

Solution 3: The number $\lim_{N\to\infty} \sum_{k=1}^{N} \frac{1}{k} - \ln(N+1)$ has a name: it's called *Euler's constant*, and is usually denoted by γ. Our picture shows that γ is a little more than $\frac{1}{2}$; you can find in any math reference book the more precise estimate $\gamma \approx 0.5772156649$. For large values of N, we have

$$\sum_{k=1}^{N} \frac{1}{k} - \ln(N+1) \approx \gamma$$

so that the partial sums of this series reach 20 when $20 \approx \ln(N+1) + \gamma$, or

$$N \approx e^{20-\gamma} - 1 \approx 272\,400\,600.$$

To check this answer, we use the Maple computer package to compute the two partial sums

$$\begin{aligned} s_{272\,400\,599} &\approx 19.999999997946378323 \\ s_{272\,400\,600} &\approx 20.000000001617442190. \end{aligned}$$