Here are some interesting problems you can solve using the techniques in §§ 6.2, 6.3, and 9.2.

1. Let r and h be positive constants. If the triangle bounded by the x-axis, the y-axis, and the lines $y = \frac{r}{h}x$ and $x = h$ is revolved about the x-axis, the resulting solid is a (right circular) cone with radius r and height h.

   Use an integral (with either disks or shells) to derive the formula for the volume of a cone with radius r and height h.

   Solution: The disks method gives

   $$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx = \frac{1}{3}\pi r^2 h.$$

   The shells method gives

   $$V = \int_0^r 2\pi y \left(h - \frac{hy}{r}\right) dy = \frac{1}{3}\pi r^2 h.$$

2. Find a formula for the surface area of a cone with radius r and height h.

   Solution: We take $f(x) = \frac{rx}{h}$, so that $f'(x) = \frac{r}{h}$. The area is given by

   $$A = \int_0^h 2\pi \sqrt{1 + \left(\frac{r}{h}\right)^2} dx = \pi r \sqrt{r^2 + h^2}.$$

3. A sphere of radius R can be viewed as a solid of revolution: take the area under the curve $y = \sqrt{R^2 - x^2}$ between $x = -R$ and $x = R$, and revolve it about the x-axis.

   You can now use an integral to derive the formula for the volume of a sphere of radius R. Stewart does this using the disk method in Example 1 of § 6.2. Try to get the same result using the shell method.

   Solution: We get

   $$V = \int_0^R 2\pi y (2\sqrt{R^2 - y^2})\, dy = \frac{4}{3}\pi R^3.$$

4. Find a formula for the surface area of a sphere of radius R.

Solution: With $f(x) = \sqrt{R^2 - x^2}$, we get

$$f'(x) = -\frac{x}{\sqrt{R^2 - x^2}}$$

so that

$$1 + (f'(x))^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

The area is given by

$$A = \int_{-R}^{R} 2\pi\sqrt{R^2 - x^2}\sqrt{\frac{R^2}{R^2 - x^2}}\, dx = 4\pi R^2.$$

5. A *bead* is a sphere with a cylindrical hole drilled through its center. A bead is determined by two dimensions: the radius of the sphere (which we will denote R) and the radius of the hole (which we will denote r). To construct a bead as a solid of revolution, take the region bounded by the curve $y = \sqrt{R^2 - x^2}$, the curve $y = -\sqrt{R^2 - x^2}$, and the line $x = r$, and revolve it about the y-axis.

Use the shell method to find a formula for the volume of a bead with sphere radius R and hole radius r.

(Your answer should be a constant multiple of $(R^2 - r^2)^{\frac{3}{2}}$. Where does this dimension appear in the bead? See § 6.3, Problem 44.)

Using the shell method, we get

$$V = \int_{r}^{R} 2\pi x(2\sqrt{R^2 - x^2})\, dx = \frac{4}{3}\pi(R^2 - r^2)^{\frac{3}{2}}.$$

Let $h = \sqrt{R^2 - r^2}$. Then h is one-half the height of the bead. That is, the length of the hole in the bead is exactly $2h$. We can rewrite the volume formula as

$$V = \frac{4}{3}\pi h^3.$$

This has a curious consequence: two beads with the same height will always have equal volumes, even if their radii are different.

6. § 6.2, Problem 59: find a formula for the volume of a torus.

Solution: Using the washer method, we get

$$\begin{aligned} V &= \int_{-r}^{r} \pi((R+\sqrt{r^2-y^2})^2-(R-\sqrt{r^2-y^2})^2)\,dy \\ &= \int_{-r}^{r} \pi(4R\sqrt{r^2-y^2})\,dy \\ &= 4\pi R\int_{-r}^{r} \sqrt{r^2-y^2}\,dy. \end{aligned}$$

The integral $\int_{-r}^{r} \sqrt{r^2-y^2}\,dy$ is the area of a semicircle of radius r, so we get

$$\int_{-r}^{r} \sqrt{r^2-y^2}\,dy = \frac{\pi r^2}{2}.$$

Substituting this into the volume expression above, we get

$$V = 4\pi R\left(\frac{\pi r^2}{2}\right) = 2\pi^2 R r^2.$$

The shell method gives

$$V = \int_{R-r}^{R+r} 2\pi x(2\sqrt{r^2-(x-R)^2})\,dx,$$

which is somewhat more difficult to evaluate. The answer is still $2\pi^2 R r^2$.