Here are some interesting problems you can solve using the techniques in §§ 6.2, 6.3, and 9.2.

1. Let $r$ and $h$ be positive constants. If the triangle bounded by the $x$-axis, the $y$-axis, and the lines $y = \frac{r}{h} x$ and $x = h$ is revolved about the $x$-axis, the resulting solid is a (right circular) cone with radius $r$ and height $h$.

Use an integral (with either disks or shells) to derive the formula for the volume of a cone with radius $r$ and height $h$.

Solution: The disks method gives

$$V = \int_{0}^{h} \pi \left( \frac{rx}{h} \right)^2 \, dx = \frac{1}{3} \pi r^2 h.$$ 

The shells method gives

$$V = \int_{0}^{r} 2\pi y \left( h - \frac{hy}{r} \right) \, dy = \frac{1}{3} \pi r^2 h.$$ 

2. Find a formula for the surface area of a cone with radius $r$ and height $h$.

Solution: We take $f(x) = \frac{rx}{h}$, so that $f'(x) = \frac{r}{h}$. The area is given by

$$A = \int_{0}^{h} 2\pi \sqrt{1 + \left( \frac{r}{h} \right)^2} \, dx = \pi r \sqrt{r^2 + h^2}.$$ 

3. A sphere of radius $R$ can be viewed as a solid of revolution: take the area under the curve $y = \sqrt{R^2 - x^2}$ between $x = -R$ and $x = R$, and revolve it about the $x$-axis.

You can now use an integral to derive the formula for the volume of a sphere of radius $R$. Stewart does this using the disk method in Example 1 of § 6.2. Try to get the same result using the shell method.

Solution: We get

$$V = \int_{0}^{R} 2\pi y(2\sqrt{R^2 - y^2}) \, dy = \frac{4}{3} \pi R^3.$$
4. Find a formula for the surface area of a sphere of radius $R$.

Solution: With $f(x) = \sqrt{R^2 - x^2}$, we get

$$f'(x) = -\frac{x}{\sqrt{R^2 - x^2}}$$

so that

$$1 + (f'(x))^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$ 

The area is given by

$$A = \int_{-R}^{R} 2\pi \sqrt{R^2 - x^2} \sqrt{\frac{R^2}{R^2 - x^2}} \, dx = 4\pi R^2.$$

5. A bead is a sphere with a cylindrical hole drilled through its center. A bead is determined by two dimensions: the radius of the sphere (which we will denote $R$) and the radius of the hole (which we will denote $r$). To construct a bead as a solid of revolution, take the region bounded by the curve $y = \sqrt{R^2 - x^2}$, the curve $y = -\sqrt{R^2 - x^2}$, and the line $x = r$, and revolve it about the $y$-axis.

Use the shell method to find a formula for the volume of a bead with sphere radius $R$ and hole radius $r$.

(Your answer should be a constant multiple of $(R^2 - r^2)^{\frac{3}{2}}$. Where does this dimension appear in the bead? See § 6.3, Problem 44.)

Using the shell method, we get

$$V = \int_{r}^{R} 2\pi x(2\sqrt{R^2 - x^2}) \, dx = \frac{4}{3}\pi (R^2 - r^2)^{\frac{3}{2}}.$$ 

Let $h = \sqrt{R^2 - r^2}$. Then $h$ is one-half the height of the bead. That is, the length of the hole in the bead is exactly $2h$. We can rewrite the volume formula as

$$V = \frac{4}{3}\pi h^3.$$ 

This has a curious consequence: two beads with the same height will always have equal volumes, even if their radii are different.

Solution: Using the washer method, we get

\[ V = \int_{-r}^{r} \pi ((R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2) \, dy \]

\[ = \int_{-r}^{r} \pi (4R \sqrt{r^2 - y^2}) \, dy \]

\[ = 4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy. \]

The integral \( \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy \) is the area of a semicircle of radius \( r \), so we get

\[ \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy = \frac{\pi r^2}{2}. \]

Substituting this into the volume expression above, we get

\[ V = 4\pi R \left( \frac{\pi r^2}{2} \right) = 2\pi^2 R r^2. \]

The shell method gives

\[ V = \int_{R-r}^{R+r} 2\pi x (2\sqrt{r^2 - (x - R)^2}) \, dx, \]

which is somewhat more difficult to evaluate. The answer is still \( 2\pi^2 R r^2 \).