Math 319 | Problem Set #1 | Due 7 February 2002

Reading: Niven, Zuckerman, and Montgomery, §§1.1, 1.2.

Exercises: Write your solutions in complete sentences. Try to do all proofs using only the material in §1.2. In particular, do not rely on the Fundamental Theorem of Arithmetic.

1. Here is a careful proof of Theorem 1.1(4): *$a|b$ and $b|a$ imply $a = \pm b$.*

Proof: Suppose $a|b$ and $b|a$. Then there are integers m and n such that $am = b$ and $bn = a$. Substituting am for b in the second equation gives $amn = a$. We can rewrite this last equation as

$$a(mn - 1) = 0.$$

Thus either $a = 0$ or $mn = 1$. If $a = 0$, then $b = 0m = 0$ and we have $a = b$, as required.

If $a \neq 0$, then $mn = 1$, so that $|m| \cdot |n| = 1$. Now $|m|$ and $|n|$ are both positive integers. We claim that $|n| = 1$. If not, then $|n| > 1$, so $|m| = 1/|n| < 1$. But there are no positive integers less than 1. The contradiction shows that $|n| = 1$, so $n = \pm 1$, and $a = \pm b$, as required. ∎

(Notice the slightly sneaky use of facts about the ordering of the integers, viewed as a subset of the real numbers.)

Now prove Theorem 1.1(5): *$a|b$, $a > 0$, $b > 0$ imply $a \leq b$.* (You will want to recall the fact that if $x \leq y$ and k is positive, then $kx \leq ky$.)

2. (§1.2, problem 24) Show that there are no integers x and y such that $(x, y) = 3$ and $x + y = 100$.

3. (§1.2, problem 26) Let s and $g > 0$ be given integers. Prove that there exist integers x and y with $x + y = s$ and $(x, y) = g$ if and only if $g|s$.

4. (Based on §1.2, problem 33) Let a and b be integers, not both zero. Show that $(a, b) = (a, b, ax + by)$ for all integers x and y.

5. (§1.2, problem 44.) Let a, b, and c be integers, and assume that a and b are not both zero. Let $g = (a, b)$. Prove that $a|bc$ if and only if $\left.\frac{a}{g}\right| c$.

6. (§1.2, problem 47.) If a and $b > 2$ are positive integers, prove that $2^a + 1$ is not divisible by $2^b - 1$.

Hints: First treat the case $a < b$. Then use an induction argument (maybe) to show that the case with $a \geq b$ can be reduced to the case $a < b$.

7. (a) Use the Euclidean algorithm to find $(37401, 5853)$.

(b) Find integers x and y such that $20437x + 8538y = 1$. (A computer spreadsheet will be helpful here. The table on p. 14 of Niven is not hard to duplicate in Excel. See the website for further hints.)

(c) Find integers x, y, and z such that $323x + 901y + 1007z = 1$. Outline an algorithm for producing the greatest common divisor of three numbers as a linear combination of the numbers.

Cultural aside:

Mathematics consists essentially of:

a) proving the obvious;
b) proving the not so obvious; and
c) proving the obviously untrue.

Mathematicians are allowed to make very heavy weather of showing what everyone already knows. For example, it took mathematicians until the 1800s to prove that $1+1=2$, and not before the late 1970s were they confident of proving that any map requires no more than four colours to make it look nice, a fact known by cartographers for centuries.

There are many not-so-obvious things which can be proved true too. Like the fact that for any group of twenty-three people, there's an even chance two or more of them share a birthday. (With groups of twins this becomes almost certain. Not quite certain, as you will of course point out; they might all have been born either side of midnight).

Mathematicians are also fond of proving things which are obviously false, like all straight lines being curved, and an engaged telephone being just as likely to be free if you ring again immediately after, as if you wait twenty minutes. They also like disproving things which are obviously true, for example that the shortest distance between two points on the earth's surface on an airline route always goes across Anchorage, Alaska.

Robert Ainsley, *Bluff Your Way in Maths*