Math 319 | Problem Set #1– Hints | 8 February 2002

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1. Be sure to use all the hypotheses. If $am = b$, you'll want to show that m is positive, then that $m \geq 1$, and then (using the fact that a is positive), that $am \geq a$.

3. This is an if-and-only-if statement, so there are really two statements to prove. One is

"Suppose we are given two integers x and y. If $g = (x, y)$ and $s = x + y$, then $g|s$."

The second (and more difficult) statement is

"Suppose we are given integers $g > 0$ and s, and suppose that $g|s$. Then we can find a pair of integers x and y such that $x + y = s$ and $(x, y) = g$."

In proving the second statement, you need to use the given numbers (s and g) to construct the numbers x and y.

5. Because g divides both a and b, we know that a/g and b/g are integers. We don't know *a priori* that any other fraction in this problem is an integer.

The fact that g is the *greatest* common divisor of a and b is crucial to the "only if" part of the proof; the most effective way to use this fact seems to be through Theorem 1.10.

6. One approach is to write a three-part proof.

(a) Lemma A: If $0 < a < b$ and $b > 2$, then $2^b - 1 > 2^a + 1$, so $2^b - 1$ cannot divide $2^a + 1$. This proof just involves messing around with inequalities and exponents.

(b) Lemma B: If $a \geq b$ and $2^b - 1 | 2^a + 1$ then $2^b - 1 | 2^{a-b} + 1$. This is an application of Theorem 1.1(3).

(c) Proof: Case 1: Suppose $a < b$. Then by Lemma A, $2^b - 1 \nmid 2^a + 1$.
Case 2: Suppose $a \geq b$, and suppose $2^b - 1 | 2^a + 1$. By the division algorithm, we can write $a = bq + r$ with $0 \leq r < b$. By repeated applications of Lemma B, we can show that

$$2^b - 1 | 2^a + 1 \quad \text{implies} \quad 2^b - 1 | 2^{a-qb} + 1,$$

that is, $2^b - 1 | 2^r + 1$ with $r < b$. But by Case 1, this is impossible. The contradiction shows that $2^b - 1 \nmid 2^a + 1$ in Case 2 as well.