Math 319 | Problem Set #2 | Due 14 February 2002

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**Reading:** NZM §1.3.

**Exercises:** Write your solutions in complete sentences.

1. (§1.3, problem 8) Let $n$ be a positive integer, and let $r$ be the integer obtained by removing the last digit from $n$ and then subtracting two times the digit just removed. (See the hint in NZM for a nice way to formalize this operation.) Prove that $7|n$ if and only if $7|r$.

2. (Based on §1.3, problem 16) Find a positive integer $n$ such that $n/2$ is a square, $n/3$ is a cube, and $n/5$ is a fifth power. Have you found the least such positive integer?

3. (Based on §1.3, problem 17)

   Let $\mathcal{P}$ be the set of pairs of twin primes greater than 2. That is, let

   $$\mathcal{P} \quad = \quad \{(3,5),(5,7),(11,13),(17,19),\ldots\}.$$

   Let $\mathcal{N}$ be the set of positive integers $n > 3$ such that $n^2 - 1$ has exactly four positive divisors. Prove that there is a one-to-one correspondence between $\mathcal{P}$ and $\mathcal{N}$.

   To do this carefully, define a map $\Phi$ from the set $\mathcal{P}$ to the set $\mathcal{N}$. Show that $\Phi$ is one-to-one and onto. (This is the meaning of "one-to-one correspondence.")

4. (Part of §1.3, problem 19.) Let $a$ and $b$ be positive integers such that $(a,b) = 1$ and $ab$ is a perfect square. Prove that $a$ and $b$ are perfect squares.

   (Don't bother with the generalization to $k^{\text{th}}$ powers.)

5. (§1.3, problem 31 – also read problem 30 and the remarks following problem 31) Prove that no polynomial $f(x)$ of degree greater than (or equal to) 1 with integral coefficients can represent a prime for every positive integer $x$.

   The hint in the back of the book gives a new level of meaning to the word "cryptic." But there's a useful idea there. Also, you may want to use the fact (from the Fundamental Theorem of Algebra) that if a polynomial $f$ of degree less than or equal to $n$ has more than $n$ distinct roots, then $f$ must be identically zero.

**6.** (Part of §1.3, problem 53) Let $\pi(x)$ denote the number of primes not exceeding $x$. Show that

$$\sum_{p \le x} \frac{1}{p} \quad = \quad \frac{\pi(x)}{x} + \int_2^x \frac{\pi(u)}{u^2}\, du$$

where the sum is taken over all primes $p$ less than or equal to $x$.

(Don't try to do the second part of the problem in the book – but do notice that we have taken another step toward the Prime Number Theorem.)

Hint: Break up the integral into a sum of integrals, each going from $n-1$ to $n$. What are the possible values of $\pi(n) - \pi(n-1)$?

**Cultural aside:**

And so it was only with the advent of pocket computers that the startling truth became finally apparent, and it was this:

*Numbers written on restaurant checks within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe.*

This single statement took the scientific world by storm. It completely revolutionized it. So many mathematical conferences got held in such good restaurants that many of the finest minds of a generation died of obesity and heart failure and the science of math was put back by years.

Slowly, however, the implications of the idea began to be understood. To begin with it had been too stark, too crazy, too much like what the man in the street would have said – "Oh, yes, I could have told you that." Then some phrases like "Interactive Subjectivity Frameworks" were invented, and everybody was able to relax and get on with it.

The small groups of monks who had taken up hanging around the major research institutes singing strange chants to the effect that the Universe was only a figment of its own imagination were eventually given a street theater grant and went away.

Douglas Adams, *The Restaurant at the End of the Universe*