Math 319 Problem Set #5 Due 28 March 2002

Exercises: Write your solutions in complete sentences.

1. (a) Compute $\varphi(m)$ for $m = 9$, $m = 25$, and $m = 49$ using the following brute-force method. Write down a list of all the integers from 1 to m (inclusive), and then cross out all the integers that aren't relatively prime to m.

(b) Make a conjecture about the value of $\varphi(p^2)$ when p is a prime. Explain why you think your conjecture is true.

2. (a) Using the same method as in Problem 1a, find $\varphi(m)$ for $m = 15$, $m = 21$, and $m = 35$.

(b) Make a conjecture about the value of $\varphi(pq)$ where p and q are primes with $p \neq q$. Explain why you think your conjecture is true.

3. The number 1013 is a prime.

(a) Using no computing power beyond a standard hand-held calculator, find $\overline{234}$ modulo 1013 (expressed as an integer between 1 and 1012 inclusive). Assume that your calculator maintains only eight digits of precision, and explain how you can work around this limitation.

(b) Solve the linear congruence $234x + 567 \equiv 251 \pmod{1013}$.

4. The number 667 is the product of the two primes 23 and 29.

(a) Find $\overline{67}$ modulo 667 (expressed as an integer between 1 and 666 inclusive).

(b) What happens when you use the same technique to try to find $\overline{46}$ modulo 667?

5. (a) Find two solutions to the congruence $x^2 + 1 \equiv 0 \pmod{1013}$. Recall that 1013 is prime. Your solutions should not be congruent to one another modulo 1013. Please do not try to do this problem by brute force.

HINT: First show that any x satisfying the given congruence also satisfies $x^4 \equiv 1 \pmod{1013}$. Fermat's theorem will then tell you where to look for solutions.

(b) Prove the following: Let p be a prime. If x and y are integers such that $x^2 \equiv y^2 \pmod{p}$, then either $x \equiv y \pmod{p}$ or $x \equiv -y \pmod{p}$.

Use this result to explain why the solutions you found in Problem 5a are the only solutions to $x^2 \equiv 1 \pmod{1013}$. (That is, any other solution must be congruent to one of your solutions modulo 1013.)

Cultural aside:

Is there such a thing as an unforgivable sin? If so, what is it and how do I commit it?

The worst sin, the act that all races and religions agree is against the laws of nature, is to attempt to divide by zero. Even computers will stop, horrified, in their tracks and chastise any operator crass enough to attempt such a thing. Division by zero may leave an indelible stain on the soul, which can be erased only by a full confession to a qualified mathematician. Even then, this sin must be atoned for in a purgatory filled with unbalanced checkbooks and tedious exercises in long division. But that's the way it has to be; otherwise, the entire system of checks and balances would fly out the window.

Why can't you divide by zero?

I can and often do divide by zero, but only after I've made the necessary preparations. First of all, I fast for forty-eight hours, consuming during that time only mildly fluoridated water. Next I don my special Mylar/Teflon division-by-zero suit. Then I put on a digitally recorded compact disc of Gregorian chants and begin with dividing very small numbers by other very small numbers. As the numbers get smaller, the sparks begin to fly. If all goes well, I take a deep breath and divide a very small number by zero. There's a flash of light, a muffled roar, and when I come to, the lab is filled with smoke and the scent of burning Mylar. So you see, you can divide by zero if you really want to. But chances are you just don't want to badly enough.

Dr. Science, *The Official Dr. Science Big Book of Science, Simplified*