

Reading: NZM §2.3

Exercises: Write your solutions in complete sentences.

1. (§2.3, problem 17) Find all solutions to the congruence

$$x^3 - 9x^2 + 23x - 15 \equiv 0 \pmod{143}.$$

(Note that $x^3 - 9x^2 + 23x - 15 = (x - 1)(x - 3)(x - 5)$.)

2. (§2.3, problem 18) Given any positive integer k , prove that there are k consecutive integers each divisible by a square greater than 1.
3. (§2.3, problem 20) Let m_1 and m_2 be arbitrary positive integers, and let $g = (m_1, m_2)$. Let a_1 and a_2 be arbitrary integers. Show that the system

$$\begin{aligned}x &\equiv a_1 \pmod{m_1} \\x &\equiv a_2 \pmod{m_2}\end{aligned}$$

has a solution if and only if $a_1 \equiv a_2 \pmod{g}$.

Show that if this condition is met, then the solution is unique modulo $[m_1, m_2]$. (That is, if x_1 and x_2 are solutions to the system, then $x_1 \equiv x_2 \pmod{[m_1, m_2]}$.)

4. Find all positive integers x such that $\varphi(x) = 48$. Try to find a reasonably organized way to do this. One way to start might be to list all the possible values of $\varphi(p^\alpha)$ for small primes p and small exponents α .
5. (§2.3, problem 40) Prove that for $n \geq 2$ the sum of all positive integers less than n and relatively prime to n is $\frac{n\varphi(n)}{2}$.

Cultural aside:

“Your hair wants cutting,” said the Hatter. He had been looking at Alice for some time with great curiosity, and this was his first speech.

“You should learn not to make personal remarks,” Alice said with some severity: “It’s very rude.”

The Hatter opened his eyes very wide on hearing this; but all he *said* was, “Why is a raven like a writing-desk?”

“Come, we shall have some fun now!” thought Alice. “I’m glad they’ve begun asking riddles – I believe I can guess that,” she added aloud.

“Do you mean that you think you can find out the answer to it?” said the March Hare.

“Exactly so,” said Alice.

“Then you should say what you mean,” the March Hare went on.

“I do,” Alice hastily replied; “at least – at least I mean what I say – that’s the same thing, you know.”

“Not the same thing a bit!” said the Hatter. “Why, you might just as well say that ‘I see what I eat’ is the same thing as ‘I eat what I see’!”

“You might just as well say,” added the March Hare, “that ‘I like what I get’ is the same thing as ‘I get what I like’!”

“You might just as well say,” added the Dormouse, which seemed to be talking in its sleep, “that ‘I breathe when I sleep’ is the same thing as ‘I sleep when I breathe’!”

“It *is* the same thing with you,” said the Hatter, and here the conversation dropped, and the party sat silent for a minute, while Alice thought over all she could remember about ravens and writing-desks, which wasn’t much.

Lewis Carroll, *Alice’s Adventures in Wonderland*