Math 319 | Problem Set #9 | Due 7 May 2002

Instructions: Write your solutions in complete sentences.

1. Find all solutions to the congruence

$$x^4 + 6x^3 + 302x^2 + 6x + 301 \equiv 0 \pmod{343}.$$

2. Let $f(x) = x^3 + 27x^2 + 80x + 49$. Solve each of the congruences

$$f(x) \equiv 0 \pmod{5}, \quad f(x) \equiv 0 \pmod{25}, \quad \text{and} \quad f(x) \equiv 0 \pmod{125}.$$

How many solutions are there to the congruence $f(x) \equiv 0 \pmod{625}$?

3. (§27, problem 6) Theorem 2.26 states that if p is prime and $f(x)$ is a polynomial with integer coefficients then the congruence $f(x) \equiv 0 \pmod{p}$ has at most n solutions, where n is the degree of the congruence.

Show that the hypothesis that p is prime is necessary. That is, for some composite number m, find an example of a congruence $f(x) \equiv 0 \pmod{m}$ with degree n and more than n roots.

4. Let $f(x)$ be a polynomial with integer coefficients, and let p be a prime. Suppose x_1 and x_2 are integers such that $x_1 \not\equiv x_2 \pmod{p^2}$, $f(x_1) \equiv f(x_2) \pmod{p^2}$, and $x_1 \equiv x_2 \pmod{p}$. Prove that

$$f'(x_1) \equiv f'(x_2) \equiv 0 \pmod{p}.$$

HINT: Use Taylor's theorem to rewrite $f(x_2) = f(x_1 + (x_2 - x_1))$.

5. (a) Consider the sequence $\{s_n\}$ given by

$$s_k = 3 + 2 \times 5 + 2 \times 5^2 + 2 \times 5^3 + \cdots + 2 \times 5^k.$$

Prove by that $2s_k - 1 = 5^{k+1}$ for each integer $k \geq 0$. (HINT: Look up "geometric series" in the index of any calculus book.) Use this result to prove that the 5-adic number $3 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots$ is equal to $\frac{1}{2}$.

(b) Find 5-adic representations of the form $5^N(d_0 + d_1 \cdot 5 + d_2 \cdot 5^2 + d_3 \cdot 5^3 + \cdots)$ (with each d_i a non-negative integer) for the numbers -2 and $\frac{1}{3}$.

Cultural aside:

The author of the bill was a physician, Edwin J. Goodman, M.D., of Solitude, Posey County, Indiana, and it was introduced in the Indiana House on January 18, 1897, by Mr. Taylor I. Record, Represenatative from Posey County. It was entitled “A bill introducing a new Mathematical truth,” and it became House Bill No. 246; copies of the bill are preserved in the Archives Division of the Indiana State Library ...

The preamble to the bill informs us that this is

> *A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted an adopted by the official action of the legislature in 1897.*

The bill consisted of three sections. Section 1 starts off like this:

> *Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle’s area is entirely wrong ...*

An “equilateral rectangle” is, of course, a square, so that the first statement does not make any sense at all; but if we give the author the benefit of the doubt and assume that this is a transcript error for “equilateral triangle,” then what Mr. Goodwin of Solitude, Posey County, had discovered in his first statement was the equivalent of $\pi = 16/\sqrt{3} = 9.2736\ldots$, which probably represents the biggest overestimate of π in the history of mathematics. ...

The bill was, perhaps symbolically, referred to the House Committee on Swamp Lands, which passed it on to the Committee of Education, and the latter reported it back to the House “with recommendation that said bill do pass.” On February 5, 1897, the House passed the learned treatise unanimously (67 to 0).

Five days later the bill went to the Senate, where it was referred, for unknown reasons, to the Committee on Temperance. The Committee on Temperance, too, reported it back to the Senate with the recommandation that it pass the bill, and it pased the first reading without comment.

Petr Beckmann, *A History of* π