Math 139 | Problem Set #6 | Due 15 April 2003

Reading: Singh, Chapter 6, Barr, §§ 4.1, 4.3

Exercises: Write your solutions clearly, remembering that they will be graded for presentation as well as correctness.

A1. (Based on Barr, §4.1, exercise 3.)

(a) Let n be an integer greater than 2. Consider the list of numbers

$$n! + 2,\ n! + 3,\ n! + 4,\ \ldots,\ n! + n.$$

Explain why all the numbers in this list are composite.

(b) Use the idea in the previous part to find a list of ten consecutive composite numbers.

(c) Use the Table of Primes on pp. 370–372 to find the *smallest* ten consecutive composite numbers.

(d) Do you think that there are 100 consecutive composite numbers? Explain your reasoning.

A2. (Based on Barr, §4.1, exercise 8.)

(a) At most how many primes will you have to divide into 11017 to determine whether 11017 is prime?

(b) Find the prime factorization of 11017.

(c) At most how many primes will you have to divide into 100,000,003 to determine whether it is prime?

A3. Suppose we have a computer that generates random eight-digit numbers, and then filters out all those that are divisible by 2 or 5, so that each time we run the program, it gives us an eight-digit number that is *not* divisible by 2 or 5.

(a) How many different outcomes are possible when we run the program?

(b) According to the Prime Number Theorem, about how many eight-digit primes are there?

(c) How many numbers will our computer program need to pick before there's a 90% probability that at least one of them is prime?

B1. (a) Use the Euclidean algorithm to find $\gcd(78993, 24513)$.

(b) Use the extended Euclidean algorithm to find two integers s and t such that $58249s + 26965t = 1$.

(c) Solve the congruences

i. $26965x \equiv 14325 \pmod{58249}$

ii. $58249x \equiv 14325 \pmod{26965}$

B2. You probably know the theorem that says a number N is divisible by 3 if and only if the sum of its decimal digits is divisible by 3. That is, if we write the $(k+1)$-digit number N as

$$N \quad = \quad d_k\, d_{k-1} \,\cdots\, d_2\, d_1\, d_0,$$

(where the d_i stand for the digits of N) then N is divisible by 3 if and only if the sum $d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0$ is divisible by 3.

Here's why that works. Writing the digits of N as above, we have

$$N \quad = \quad d_0 + 10 \times d_1 + 100 \times d_2 + \cdots + 10^{k-1} \times d_{k-1} + 10^k \times d_k. \tag{1}$$

Now we recall that $10 \equiv 1 \pmod 3$, $100 \equiv 1 \pmod 3$, $1000 \equiv 1 \pmod 3$, and so on. That is, each positive power of 10 is congruent to 1 modulo 3.

So equation 1, written as a mod-3 congruence, becomes

$$\begin{aligned} N &\equiv d_0 + 1 \times d_1 + 1 \times d_2 \times + \cdots + 1 \times d_{k-1} + 1 \times d_k \pmod 3 \\ &\equiv d_0 + d_1 + d_2 + \cdots + d_{k-1} + d_k \pmod 3. \end{aligned}$$

Thus the sum of the digits of N is divisible by 3 (that is, congruent to 0 modulo 3) if and only if N itself divisible by 3.

Here's a similar test for divisibility by 11. Given a number

$$N \quad = \quad d_k\, d_{k-1} \,\cdots\, d_2\, d_1\, d_0,$$

adjoin a leading zero if necessary so that the number of digits is even. Use the digits of N in pairs to form a sequence of two-digit numbers. Then N is divisible by 11 if and only if the sum of these two-digit numbers is divisible by 11. That is, N is divisible by 11 if and only if

$$(d_k\, d_{k-1}) + (d_{k-2}\, d_{k-3}) + \cdots + (d_3\, d_2) + (d_1\, d_0)$$

is divisible by 11.

(a) Explain why this divisibility test works.

(b) Use this test to find the smallest number x for which

$$10200111100011010010 + x$$

is divisible by 11.