A1. (Based on Barr, §4.1, exercise 3.)

(a) Let n be an integer greater than 2. Consider the list of numbers

$$n! + 2,\ n! + 3,\ n! + 4,\ \ldots,\ n! + n.$$

Explain why all the numbers in this list are composite.

Solution: We know that $n!$ is divisible by all the numbers from 2 up to n.

It follows that $n!+2$ is divisible by 2, since both $n!$ and 2 are divisible by 2. Also, we know that $n! + 2 > 2$. Since $n! + 2 > 2$ and has 2 as a factor, we find that $n! + 2$ must be composite.

Similarly, for each j between 2 and n (inclusive), the number $n! + j$ is divisible by j and greater than j, so it must have j as a (proper) factor. Thus $n! + j$ is composite.

(b) Use the idea in the previous part to find a list of ten consecutive composite numbers.

Solution: We start with $11! = 39916800$. Then we know the following numbers are all composite:

$$\begin{aligned} 11! + 2 &= 39916802 \\ 11! + 3 &= 39916803 \\ 11! + 4 &= 39916804 \\ \vdots \quad & \qquad \vdots \\ 11! + 11 &= 39916811. \end{aligned}$$

(c) Use the Table of Primes on pp. 370–372 to find the *smallest* ten consecutive composite numbers.

Solution: According to the table, the first run of 10 consecutive composite numbers begins at 114. The numbers

$$114,\ 115,\ 116,\ 117,\ 118,\ 119,\ 120,\ 121,\ 122,\ 123$$

are all composite. (So are 124, 125, and 126.)

(d) Do you think that there are 100 consecutive composite numbers? Explain your reasoning.

Solution: The numbers $101! + 2$ through $101! + 101$ form a sequence of 100 consecutive composite numbers.

A2. (Based on Barr, §4.1, exercise 8.)

(a) At most how many primes will you have to divide into 11017 to determine whether 11017 is prime?

Solution: We need to check the primes that are less than or equal to $\sqrt{11017} \approx 104.96$. So we need to check the primes

$$2,\ 3,\ 5,\ \ldots,\ 103.$$

There are 27 primes in this list.

(b) Find the prime factorization of 11017.

Solution: We try the primes in the list above one at a time, and find that 11017 is divisible by 23. In fact,

$$11017 \quad = \quad 23 \times 479.$$

Moreover, the number 479 appears in our list of primes, so we have found the prime factorization of 11017.

(c) At most how many primes will you have to divide into 100,000,003 to determine whether it is prime?

Solution: We find that $\sqrt{100,000,003} \approx 10,000$, so we'd need to divide this number by all the primes less than $10,000$. From the primes table in the book, we determine that there are 1229 primes in this range.

A3. Suppose we have a computer that generates random eight-digit numbers, and then filters out all those that are divisible by 2 or 5, so that each time we run the program, it gives us an eight-digit number that is *not* divisible by 2 or 5.

(a) How many different outcomes are possible when we run the program?

Solution: A number is divisible by 2 if its last digit is even, and divisible by 5 if its last digit is 0 or 5. Numbers that are not divisible by 2 or 5 are those whose last digit is 1, 3, 7, or 9.

The first digit of an eight-digit number must be between 1 and 9, inclusive.

Each of the intermediate digits can be anything from 0 to 9.

To form an eight-digit number that's not divisible by 2 or 5, we need to

choose a first digit (9 possibilites)
choose a second digit (10 possibilities)
choose a third digit (10 possibilities)
...
choose a seventh digit (10 possibilities)
choose a last digit (4 possibilities).

The total number of ways we can do this is

$$9 \times 10^6 \times 4 \quad = \quad 36,000,000.$$

Solution: (alternate) There are $99999999 - 9999999 = 90000000$ different eight-digit numbers. Of these, the number that are divisble by 4 is

$$90000000/2 \quad = \quad 45000000.$$

The number that are divisible by 5 is

$$90000000/5 \quad = \quad 18000000.$$

There are $90000000/10 = 9000000$ eight-digit numbers that are divisible by both 2 and 5 (that is, they are divisible by 10). To count the number of eight-digit numbers that are not divisible by 2 or by 5, we take 90000000 and subtract 45000000, then subtract 18000000. Then we have to add back 9000000, because we've subtracted the multiples of ten twice. There are

$$90000000 - 45000000 - 18000000 + 9000000 = 36000000$$

eight-digit numbers that are not divisible by 2 or 5.

(b) According to the Prime Number Theorem, about how many eight-digit primes are there?

Solution: There are about

$$\frac{99999999}{\ln 99999999} - \frac{9999999}{\ln 9999999} \quad \approx \quad 4808260$$

eight-digit primes.

(c) How many numbers will our computer program need to pick before there's a 90% probability that at least one of them is prime?

Solution: The probability that we get a prime number on any particular trial is about

$$\frac{480260}{36000000} \approx 0.13356,$$

so the probability of getting a composite number on any particular trial is about

$$1 - 0.13356 = 0.86644.$$

The probability of getting n composite numbers on n consecutive trials is about 0.86644^n, so we need to find the least n such that $0.86644^n < 0.1$. This turns out to be $n = 17$.

B1. (a) Use the Euclidean algorithm to find $\gcd(78993, 24513)$.

Solution: Taking residues in each line, we have

$$\begin{aligned}
\gcd(78993, 24513) &= \gcd(24513, 5454) \\
&= \gcd(5454, 2697) \\
&= \gcd(2697, 60) \\
&= \gcd(60, 57) \\
&= \gcd(57, 3) \\
&= \gcd(3, 0).
\end{aligned}$$

So $\gcd(78993, 24513) = 3$.

(b) Use the extended Euclidean algorithm to find two integers s and t such that $58249s + 26965t = 1$.

Solution: By the usual division algorithm, we have

$$\begin{array}{rcccrcl}
58249 & = & 2 & \times & 26965 & + & 4319 \\
26965 & = & 6 & \times & 4319 & + & 1051 \\
4319 & = & 4 & \times & 1051 & + & 115 \\
1051 & = & 9 & \times & 115 & + & 16 \\
115 & = & 7 & \times & 16 & + & 3 \\
16 & = & 5 & \times & 3 & + & 1
\end{array}$$

Now we do the back substitution to get

$$\begin{aligned}
1 &= 16 - 5 \times 3 \\
&= 16 - 5(115 - 7 \times 16) \\
&= -5 \times 115 + 36 \times 16 \\
&= -5 \times 115 + 36(1051 - 9 \times 115) \\
&= 36 \times 1051 - 329 \times 115 \\
&= 36 \times 1051 - 329(4319 - 4 \times 1051) \\
&= -329 \times 4319 + 1352 \times 1051 \\
&= -329 \times 4319 + 1352(26965 - 6 \times 4319) \\
&= 1352 \times 26965 - 8441 \times 4319 \\
&= 1352 \times 26965 - 8441(58249 - 2 \times 26965) \\
&= -8441 \times 58249 + 18234 \times 26965.
\end{aligned}$$

The solution is

$$-8441 \times 58249 + 18234 \times 26965 = 1$$

(c) Solve the congruences

i. $26965x \equiv 14325 \pmod{58249}$

Solution: From above, we know that $26965 \times 18234 \equiv 1 \pmod{58249}$, so we multiply both sides of the given congruence by 18234 to get

$$x \equiv 18234 \times 14325 \equiv 13534 \pmod{58249}.$$

ii. $58249x \equiv 14325 \pmod{26965}$

Solution: From above, we know that $58249 \times 8441 \equiv -1 \pmod{26965}$, so it follows that -8441 is the multiplicative inverse of 58249 modulo 26965. We have

$$-8441 \equiv 18524 \pmod{26965}.$$

We multiply both sides of the given congruence by 18524 to get

$$x \equiv 18524 \times 14325 \equiv 20700 \pmod{26965}$$

B2. You probably know the theorem that says a number N is divisible by 3 if and only if the sum of its decimal digits is divisible by 3. That is, if we write the $(k+1)$-digit number N as

$$N \;=\; d_k\, d_{k-1}\, \cdots\, d_2\, d_1\, d_0,$$

(where the d_i stand for the digits of N) then N is divisible by 3 if and only if the sum $d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0$ is divisible by 3.

Here's why that works. Writing the digits of N as above, we have

$$N \;=\; d_0 + 10 \times d_1 + 100 \times d_2 + \cdots + 10^{k-1} \times d_{k-1} + 10^k \times d_k. \qquad (1)$$

Now we recall that $10 \equiv 1 \pmod 3$, $100 \equiv 1 \pmod 3$, $1000 \equiv 1 \pmod 3$, and so on. That is, each positive power of 10 is congruent to 1 modulo 3.

So equation 1, written as a mod-3 congruence, becomes

$$\begin{aligned} N &\equiv d_0 + 1 \times d_1 + 1 \times d_2 \times + \cdots + 1 \times d_{k-1} + 1 \times d_k \pmod 3 \\ &\equiv d_0 + d_1 + d_2 + \cdots + d_{k-1} + d_k \pmod 3. \end{aligned}$$

Thus the sum of the digits of N is divisible by 3 (that is, congruent to 0 modulo 3) if and only if N itself divisible by 3.

Here's a similar test for divisibility by 11. Given a number

$$N \;=\; d_k\, d_{k-1}\, \cdots d_2\, d_1\, d_0,$$

adjoin a leading zero if necessary so that the number of digits is even. Use the digits of N in pairs to form a sequence of two-digit numbers. Then N is divisible by 11 if and only if the sum of these two-digit numbers is divisible by 11. That is, N is divisible by 11 if and only if

$$(d_k\, d_{k-1}) + (d_{k-2}\, d_{k-3}) + \cdots + (d_3\, d_2) + (d_1\, d_0)$$

is divisible by 11.

(a) Explain why this divisibility test works.

Solution: Consider the equations

$$\begin{array}{rcrcl} 100 &=& 9 \times 11 &+& 1 \\ 10000 &=& 909 \times 11 &+& 1 \\ 1000000 &=& 90909 \times 11 &+& 1 \\ \vdots & & \vdots & & \end{array}$$

The pattern shows that $100^k \equiv 1 \pmod{11}$ for each positive integer k.
Now if we write a number N as a string of its digits,

$$N = d_k\, d_{k-1}\, d_{k-2} \cdots d_2\, d_1\, d_0,$$

then we have

$$\begin{aligned} N &= (d_1\, d_0) + 100 \times (d_3\, d_2) + 100^2 \times (d_5\, d_4) + \cdots + 100^{k-1} \times (d_k\, d_{k-1}) \\ &\equiv (d_1\, d_0) + 1 \times (d_3\, d_2) + 1 \times (d_5\, d_4) + \cdots + 1 \times (d_k\, d_{k-1}) \pmod{11} \\ &\equiv (d_1\, d_0) + (d_3\, d_2) + (d_5\, d_4) + \cdots + (d_k\, d_{k-1}) \pmod{11}. \end{aligned}$$

This shows that N is divisible by 11 if and only if the sum

$$(d_k\, d_{k-1}) + (d_{k-2}\, d_{k-3}) + \cdots + (d_3\, d_2) + (d_1\, d_0)$$

is divisible by 11.

(b) Use this test to find the smallest number x for which

$$10200111100011010010 + x$$

is divisible by 11.

Solution: We separate the digits of the given number into pairs as

$$10200111100011010010 = 10\ 20\ 01\ 11\ 10\ 00\ 11\ 01\ 00\ 10.$$

We find that

$$10 + 20 + 01 + 11 + 10 + 00 + 11 + 01 + 00 + 10 = 74.$$

The least positive number we can add to this sum to make it into a multiple of 11 is $x = 3$. The number 10200111100011010013 is divisible by 11.