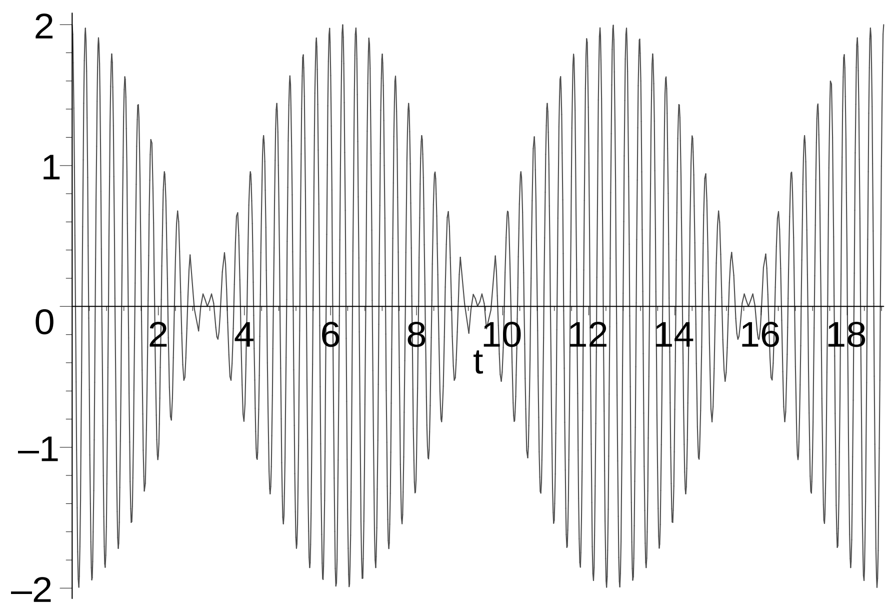
Notes: Beats occur whenever two sinusoid functions with approximately the same frequency and amplitude are added together. Here is a plot showing the function $y(t) = \cos(20t) + \cos(21t)$ for $0 \le t \le 6\pi$.



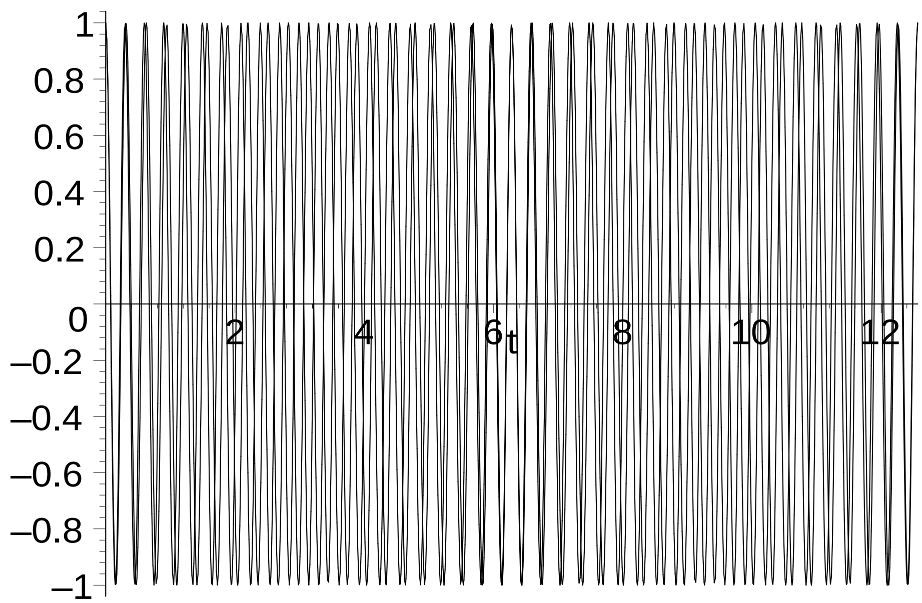
The sum of these two cosine functions is a function that osciallates at a frequency somewhere between 20 and 21 radians per unit time, with an amplitude envelope that rises and falls at the rate of one "beat" every π units of time.

We can understand this behavior in (at least) two ways:

1. The two cosine functions start out in phase with one another, so that adding them together is almost like doubling one of them. They are both positive at the same times, and both negative at the same times. Due to the difference in frequencies, however, the relative phase angle between the two cosine functions slowly increases. Eventually, the relative phase angle

reaches π, so that one of the cosine functions is very nearly the negative of the other one, and their sum is very nearly zero. The relative phase angle continues to increase, through 2π (where the two functions again reinforce one another), then 3π (where they again cancel each other out), and so on.

Here is a picture of the two functions $\cos(20t)$ and $\cos(21t)$. Note how they slip in and out of phase with one another.



2. We can use trigonometric identities to help quantify this phenomenon. Consider two functions $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$. Let

$$A = \frac{\omega_2 + \omega_1}{2} \quad \text{and} \quad B = \frac{\omega_2 - \omega_1}{2}.$$

With these definitions, we note that $A + B = \omega_2$ and $A - B = \omega_1$. Then, using the addition formula for cosine, we have

$$
\begin{aligned}
\cos(\omega_1 t) + \cos(\omega_2 t) &= \cos(At - Bt) + \cos(At + Bt) \\
&= \cos At \cos Bt + \sin At \sin Bt + \cos At \cos Bt - \sin At \sin Bt \\
&= 2\cos At \cos Bt \\
&= 2\cos Bt \cos At \\
&= 2\cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_2 + \omega_1}{2} t\right).
\end{aligned}
$$

If ω_1 and ω_2 are close together, then the term $2\cos(((\omega_2 - \omega_1)/2)t)$ is a slowly-varying sinusoid, which may be viewed as a slowly-varying coefficient controlling the amplitude of the much higher-frequency term $\cos(((\omega_2 + \omega_1)/2)t)$.

This explains the beats in the picture above. With the numbers there, the beat frequency is $(21 - 20)/2 = 1/2$, so the amplitude of the sum $\cos(20t) + \cos(21t)$ varies as $|2\cos(t/2)|$. It is close to 2 when $t = 0$, close to zero when $t = \pi$, close to 1 again when $t = 2\pi$, and so on.

Beats turn up in the study of mass-and-spring systems when an undamped system has a natural frequency ($\omega_0 = \sqrt{k/m}$) and is driven at a frequency ω that is close, but not equal, to ω_0. The IVP for such a system takes the form

$$mu'' + ku = F_0 \cos(\omega t); \quad y(0) = y_0, \quad y'(0) = y_0'.$$

The complementary part of the solution is a sinusoid with frequency ω_0, and the driven part of the solution is a sinusoid with frequency ω. The sum of these two will exhibit beats – the amplitude will rise and fall with frequency $|\omega - \omega_0|/2$.

Exercises:

(These are just for practice, and are not to be turned in.)

1. Use trigonometric identities to show that $\sin(\omega_1 t) + \sin(\omega_2 t)$ and $\sin(\omega_1 t) + \cos(\omega_2 t)$ can also be written as products of sinusoids where one factor has a low frequency and one has a higher frequency.

2. Plot the functions $\cos(20t) + \cos(21t)$ and $2\cos(t/2)$ on the same axes to convince yourself that the latter really is the amplitude envelope of the former.

3. In our examples above, the two functions we added together had identical amplitudes. What happens if the amplitudes are different? Plot the function $\cos(10t) + 2\cos(11t)$.

Do you see beats? What is the beat frequency? Use trigonometric identities to rewrite $\cos(10t) + 2\cos(11t)$ in a way that explains the beats and gives their frequency.

4. Consider the IVP

$$y'' + 25y = \sin 6t; \quad y(0) = 0, \quad y'(0) = 1.$$

   Without solving the IVP, predict the frequency at which the amplitude of the solution will rise and fall. Then solve the IVP, plot your solution, and see if your prediction was correct.

5. Try to observe aural beats in some everyday setting. Two musical notes of very nearly the same pitch will usually beat against one another. Two appliances with motors running an slightly different speeds sometimes produce beats.