Math 333 | Solution
Hour Exam I | 27 –28 February 2003

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1. Consider the differential equation $y' + 2ty = 3t$.

   (a) Use the method of integrating factors to find the general solution.

   Solution: The integrating factor is $\exp\left(\int 2t\,dt\right)$, which evaluates to e^{t^2}. Multiplying through by e^{t^2}, we get

   $$e^{t^2}y' + 2te^{t^2}y = 3t$$

   so that

   $$\frac{d}{dt}\left(e^{t^2}y\right) = 3te^{t^2}.$$

   We integrate both sides to get

   $$e^{t^2}y = \frac{3}{2}e^{t^2} + C$$

   The general solution is

   $$y(t) = \frac{3}{2} + Ce^{-t^2}.$$

   (b) Describe the long-term behavior (that is, as $t \to \infty$) of solutions to this differential equation.

   Solution: As $t \to \infty$, every solution approaches $\frac{3}{2}$.

2. Solve the initial value problem $e^{3x}\frac{dy}{dx} = \frac{1}{y-4}$; $\quad y(0) = 1$.

   Solution: The equation is separable. We write

   $$(y-4)dy = e^{-3x}\,dx$$

   and integrate both sides to get

   $$\frac{y^2}{2} - 4y = -\frac{1}{3}e^{-3x} + C.$$

At this point, we can use the initial condition to find C. We have

$$-\frac{7}{2} = -\frac{1}{3} + C$$

so $C = -\dfrac{19}{6}$.

We multiply both sides of our equation by 2 to get

$$y^2 - 8y = -\frac{2}{3}e^{-3x} - \frac{19}{3}.$$

Next we complete the square on the left to get

$$\begin{aligned} y^2 - 8y + 16 &= -\frac{2}{3}e^{-3x} - \frac{19}{3} + 16 \\ (y-4)^2 &= \frac{29 - 2e^{-3x}}{3} \end{aligned}$$

so that

$$y = 4 \pm \sqrt{\frac{29 - 2e^{-3x}}{3}}.$$

Using the initial condition again, we select the negative square root, so that the solution is

$$y(x) = 4 - \sqrt{\frac{29 - 2e^{-3x}}{3}}.$$

3. For what values of r is e^{rt} a solution to the differential equation $y''' - 5y' = 0$?

   Solution: We have

$$\begin{aligned} y'(t) &= re^{rt} \\ y''(t) &= r^2e^{rt} \\ y'''(t) &= r^3e^{rt}, \end{aligned}$$

so that

$$y''' - 5y' = (r^3 - 5r)e^{rt}.$$

For this to be the zero function, we must have $r^3 - 5r = 0$. That is, $r(r^2 - 5) = 0$, so $r = 0$ or $r = \pm\sqrt{5}$.

4. An investor opens a retirement account in year $t = 0$ with an initial deposit of \$4000. Anticipating salary increases through her career, she plans to deposit $5000 + 250t$ dollars in the account in year t. Suppose the deposits are made continuously, and the account pays continuously-compounded interest with a rate that fluctuates according to the function $5 + 10\sin(2\pi t/10)$ percent per year, where t is in years.

   Write an initial value problem for $V(t)$, the value of the account at time t. Don't try to solve it.

   Solution: The rate at which V is increasing is equal to the sum two quantities: one is the rate at which money is being deposited, the other is the interest rate multiplied by V itself. At $t = 0$, we know that V is 4000.

   Writing this in mathematical notation, we have We have

$$V'(t) = 5000 + 250t + \frac{5 + 10\sin(2\pi t/10)}{100} V(t); \quad V(0) = 4000.$$

5. Consider the initial value problem $y' = f(t, y)$, $y(0) = 1$, where

$$f(t, y) \quad = \quad \frac{y^3 - 2y}{t^2 + 1}.$$

   Using Euler's method, fill in the missing numbers in the following table. Give each value to four decimal places.

   Solution:

n	t_n	y_n	$f(t_n, y_n)$
0	0	1.0000	-1.0000
1	0.2	0.8000	-1.0462
2	0.4	0.5908	-0.8408
3	0.6	0.4226	

6. Consider again the differential equation

$$y' = \frac{y^3 - 2y}{t^2 + 1}.$$

Although this equation is not autonomous, it does have equilibrium solutions. Determine the three equilibrium solutions, and classify each as asymptotically stable or unstable.

Solution: The equilibrium solutions occur when $y' = 0$, and that happens exactly when $y^3 - 2y = 0$. We factor $y^3 - 2y$ to get

$$y(y^2 - 2) \;\; = \;\; 0,$$

so the equilibrium solutions occur at $y = 0$ and $y = \pm\sqrt{2}$. Since the denominator is positive for all values of t, the sign of y' is always the same as the sign of $y^3 - 2y$. That is, y' is negative for $y < -\sqrt{2}$ and $0 < y < \sqrt{2}$ and positive for $-\sqrt{2} < y < 0$ and $y > \sqrt{2}$. From these observations, we conclude that $y = 0$ is an asymptotically stable equilibrium and $y = \pm\sqrt{2}$ are unstable equilibria.