1. Consider the differential equation $y' + 2ty = 3t$.

   (a) Use the method of integrating factors to find the general solution.

   Solution: The integrating factor is $\exp\left(\int 2t \, dt\right)$, which evaluates to $e^{t^2}$. Multiplying through by $e^{t^2}$, we get
   
   $$e^{t^2}y' + 2te^{t^2}y = 3t$$
   
   so that
   
   $$\frac{d}{dt}(e^{t^2}y) = 3te^{t^2}.$$ 
   
   We integrate both sides to get
   
   $$e^{t^2}y = \frac{3}{2}e^{t^2} + C.$$ 
   
   The general solution is
   
   $$y(t) = \frac{3}{2} + Ce^{-t^2}.$$ 

   (b) Describe the long-term behavior (that is, as $t \to \infty$) of solutions to this differential equation.

   Solution: As $t \to \infty$, every solution approaches $\frac{3}{2}$.

2. Solve the initial value problem $e^{3x} \frac{dy}{dx} = \frac{1}{y - 4}; \quad y(0) = 1$.

   Solution: The equation is separable. We write
   
   $$(y - 4)dy = e^{-3x} \, dx$$
   
   and integrate both sides to get
   
   $$\frac{y^2}{2} - 4y = -\frac{1}{3}e^{-3x} + C.$$
At this point, we can use the initial condition to find $C$. We have

$$\frac{-7}{2} = \frac{1}{3} + C$$

so $C = -\frac{19}{6}$.

We multiply both sides of our equation by 2 to get

$$y^2 - 8y = -\frac{2}{3}e^{-3x} - \frac{19}{3}.$$  

Next we complete the square on the left to get

$$y^2 - 8y + 16 = -\frac{2}{3}e^{-3x} - \frac{19}{3} + 16$$

so that

$$(y - 4)^2 = \frac{29 - 2e^{-3x}}{3}.$$  

Using the initial condition again, we select the negative square root, so that the solution is

$$y(x) = 4 - \sqrt{\frac{29 - 2e^{-3x}}{3}}.$$  

3. For what values of $r$ is $e^{rt}$ a solution to the differential equation $y''' - 5y' = 0$?

Solution: We have

$$y'(t) = re^{rt}$$
$$y''(t) = r^2e^{rt}$$
$$y'''(t) = r^3e^{rt},$$

so that

$$y''' - 5y' = (r^3 - 5r)e^{rt}.$$  

For this to be the zero function, we must have $r^3 - 5r = 0$. That is, $r(r^2 - 5) = 0$, so $r = 0$ or $r = \pm\sqrt{5}$. 
4. An investor opens a retirement account in year $t = 0$ with an initial deposit of $4000. Anticipating salary increases through her career, she plans to deposit $5000 + 250t$ dollars in the account in year $t$. Suppose the deposits are made continuously, and the account pays continuously-compounded interest with a rate that fluctuates according to the function $5 + 10 \sin(2\pi t/10)$ percent per year, where $t$ is in years.

Write an initial value problem for $V(t)$, the value of the account at time $t$. Don’t try to solve it.

Solution: The rate at which $V$ is increasing is equal to the sum two quantities: one is the rate at which money is being deposited, the other is the interest rate multiplied by $V$ itself. At $t = 0$, we know that $V$ is 4000.

Writing this in mathematical notation, we have

$$V'(t) = 5000 + 250t + \frac{5 + 10 \sin(2\pi t/10)}{100} V(t); \quad V(0) = 4000.$$ 

5. Consider the initial value problem $y' = f(t, y)$, $y(0) = 1$, where

$$f(t, y) = \frac{y^3 - 2y}{t^2 + 1}.$$ 

Using Euler’s method, fill in the missing numbers in the following table. Give each value to four decimal places.

Solution:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_n$</th>
<th>$y_n$</th>
<th>$f(t_n, y_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.0000</td>
<td>-1.0000</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.8000</td>
<td>-1.0462</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.5908</td>
<td>-0.8408</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.4226</td>
<td></td>
</tr>
</tbody>
</table>

6. Consider again the differential equation

$$y' = \frac{y^3 - 2y}{t^2 + 1}.$$
Although this equation is not autonomous, it does have equilibrium solutions. Determine the three equilibrium solutions, and classify each as asymptotically stable or unstable.

Solution: The equilibrium solutions occur when \( y' = 0 \), and that happens exactly when \( y^3 - 2y = 0 \). We factor \( y^3 - 2y \) to get

\[
y(y^2 - 2) = 0,
\]

so the equilibrium solutions occur at \( y = 0 \) and \( y = \pm \sqrt{2} \). Since the denominator is positive for all values of \( t \), the sign of \( y' \) is always the same as the sign of \( y^3 - 2y \). That is, \( y' \) is negative for \( y < -\sqrt{2} \) and \( 0 < y < \sqrt{2} \) and positive for \( -\sqrt{2} < y < 0 \) and \( y > \sqrt{2} \). From these observations, we conclude that \( y = 0 \) is an asymptotically stable equilibrium and \( y = \pm \sqrt{2} \) are unstable equilibria.