Math 333 Solution
Hour Exam II 3–4 April 2003

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1. Let $y_1 = t\sin 2t$ and $y_2 = t\cos 2t$. Compute $W(y_1, y_2)$.

   Solution: We have

$$\begin{aligned} y_1' &= \sin 2t + 2t\cos 2t \\ y_2' &= \cos 2t - 2t\sin 2t \end{aligned}$$

   so that

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} t\sin 2t & \sin 2t + 2t\cos 2t \\ t\cos 2t & \cos 2t - 2t\sin 2t \end{vmatrix} \\ &= t\sin 2t\cos 2t - 2t^2\sin^2 2t - t\sin 2t\cos 2t - 2t^2\cos^2 2t \\ &= -2t^2. \end{aligned}$$

2. Find the general solution to each of the following differential equations.

   (a) $y'' + 3y' - 10y = 0$

   Solution: The characteristic equation is $r^2 + 3r - 10 = 0$. The roots of the characteristic equation are -2 and 5, so the general solution to the differential equation is

$$y = c_1 e^{-2t} + c_2 e^{5t}.$$

   (b) $y'' - 4y' + 11y = 0$

   Solution: The characteristic equation is $r^2 - 4r + 11 = 0$. The roots of the characteristic equation are $2 \pm i\sqrt{7}$, so the general solution to the differential equation is

$$y = e^{2t}(c_1 \cos t\sqrt{7} + c_2 \sin t\sqrt{7}).$$

(c) $y'' + 6y' + 9y = 0$

Solution: The characteristic equation is $r^2 + 6r + 9 = 0$. The roots of the characteristic equation are both -3, so the general solution to the differential equation is

$$y = (c_1 t + c_2)e^{-3t}.$$

3. Find a particular solution Y to each of the following differential equations.

(a) $y'' + 4y = \cos 2t$

Solution: Because the functions $\cos 2t$ and $\sin 2t$ are in the complementary solution, we guess a particular solution has the form $Y = At\cos 2t + Bt\sin 2t$. We get

$$\begin{aligned} Y' &= A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t \\ Y'' &= -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t. \end{aligned}$$

Thus the equation $Y'' + 4Y = \cos 2t$ becomes

$$\begin{aligned} \cos 2t &= -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t + 4At\cos 2t + 4Bt\sin 2t \\ &= -4A\sin 2t + 4B\cos 2t. \end{aligned}$$

Equating coefficients gives $-4A = 0$ and $4B = 1$, so that $A = 0$ and $B = \dfrac{1}{4}$. Our particular solution is

$$Y = \frac{1}{4}t\sin 2t.$$

(b) $y'' + y' - 6y = te^{2t}$

Soltution: The characteristic equation, $r^2 + r - 6 = 0$, happens to have $r = 2$ as a root, so we know that e^{2t} is in the complementary solution for this differential equation. Our guess for a particular solution is thus $Y = (At^2 + Bt)e^{2t}$. We get

$$\begin{aligned} Y' &= (2At + B)e^{2t} + 2(At^2 + Bt)e^{2t} \\ &= (2At^2 + 2(A+B)t + B)e^{2t} \\ Y'' &= (4At + 2(A+B))e^{2t} + 2(2At^2 + 2(A+B)t + B)e^{2t} \\ &= (4At^2 + (8A + 4B)t + (2A + 4B))e^{2t}. \end{aligned}$$

The equation $te^{2t} = Y'' + Y' - 6Y$ becomes

$$te^{2t} = ((10A + 6B - 6B)t + (2A + 5B))e^{2t}.$$

Equating coefficients, we get $10A = 1$ and $2A + 5B = 0$. The solution to the system is $A = \frac{1}{10}$ and $B = -\frac{1}{25}$. Our particular solution to the differential equation is

$$Y = \left(\frac{1}{10}t^2 - \frac{1}{25}t\right)e^{2t}.$$

4. Solve the initial value problem

$$y'' + y' = t + 2; \quad y(-1) = 2, \quad y'(-1) = -1.$$

Solution: The characteristic equation is $r^2 + r = 0$, which has roots $r = 0$ and $r = -1$. The complementary solution is thus

$$y_c = c_1 + c_2 e^{-t}.$$

The particular solution will have the form

$$Y = At^2 + Bt.$$

From this we get

$$\begin{aligned} Y' &= 2At + B \\ Y'' &= 2A. \end{aligned}$$

Plugging these into the given equation yields

$$2A + 2At + B = t + 2.$$

Equating coefficients, we get the system

$$\begin{aligned} 2A &= 1 \\ 2A + B &= 2 \end{aligned}$$

The solution is $A = \dfrac{1}{2}$ and $B = 1$. The general solution to the differential equation has the form

$$y = \frac{1}{2}t^2 + t + c_1 + c_2 e^{-t}.$$

From this we get

$$y' = t + 1 - c_2 e^{-t}.$$

The initial conditions give us the system

$$\begin{aligned} 2 &= \frac{1}{2} - 1 + c_1 + c_2 e \\ -1 &= -1 + 1 - c_2 e. \end{aligned}$$

The solution to the system is $c_2 = \dfrac{1}{e}$, $c_1 = \dfrac{3}{2}$. The solution to the IVP is

$$y = \frac{3}{2} + e^{-t-1} + \frac{1}{2}t^2 + t.$$

5. A spring has a natural length of 15 cm. A mass of 150 gm hanging from the spring causes it to stretch an additional 5 cm. The mass is attached to a damping mechanism that exerts a force of 500 dynes when the mass moves at 1 cm/sec.

   From its rest position, the mass is struck with enough force to give it an initial velocity (downward) of 100 cm/sec, and then is allowed to move freely.

   Assume that gravitational acceleration is equal to 980 cm/sec^2.

   (a) Set up an initial value problem for $u(t)$, the position of the mass as a function of time.

   Solution: We need to find the spring constant. The mass pulls on the spring with a force equal to 150 gm times 980 cm/sec^2, which is 147,000 dynes. The spring constant is thus

$$\frac{147,000 \text{ dynes}}{5 \text{ cm}} = \frac{29,400 \text{ dynes}}{\text{cm}}.$$

   The damping coefficient is essentially given; it's

$$\frac{500 \text{ dynes-sec}}{\text{cm}}.$$

The initial conditions are $u(0) = 0$ and $u'(0) = 100$.
The initial value problem is

$$150u'' + 500u' + 29,400u = 0; \quad u(0) = 0, \quad u'(0) = 100.$$

(b) Is the system underdamped, overdamped, or critically damped?

(c) How much damping would be required for the system to be critically damped?

6. Consider the linear differential equation

$$t^2y'' + (2t^2 - t)y' - (2t - 1)y = 0$$

with $t > 0$. It is easy to check that $y_1 = t$ is one solution to the differential equation. Use the method of reduction of order to find a second solution that is not a scalar multiple of y_1.

Solution: We posit a second solution $y_2 = tv$, where v is a function of t. We get

$$\begin{aligned} y_2' &= v + tv' \\ y_2'' &= 2v' + tv'' \end{aligned}$$

Substituting these expressions into the given differential equation, we get

$$\begin{aligned} 0 &= t^2(2v' + tv'') + (2t^2 - t)(v + tv') - (2t - 1)tv \\ &= 2t^2v' + t^3v'' + 2t^2v + 2t^3v' - tv - t^2v' - 2t^2v + tv \\ &= t^3v'' + (2t^3 + t^2)v'. \end{aligned}$$

This is separable; we get

$$\frac{v''}{v'} = -2 - \frac{1}{t}.$$

Integrating both sides, we get

$$\ln v' = -2t + -\ln t + c_1$$

for some constant c_1. Solving for v', we get

$$v' = c_2t^{-1}e^{-2t}$$

Since we don't know how to integrate this (the antiderivative can't be expressed in terms of elementary functions), the best we can do is to say that $v = \int t^{-1}e^{-2t}\,dt$. Then our second solution y_2 is tv. That is, we get

$$y_2 \;=\; t\int t^{-1}e^{-2t}\,dt.$$