Math 333 Solution
Hour Exam III 2 May 2003

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1. Consider the function $y(t) = 3\cos(22t) + 5\cos(24t)$. Since this function is the sum of two sinusoids whose frequencies are close together, it will exhibit beats.

   (a) If $y(t) = A\cos(\alpha t)\cos(\beta t) + B\cos(24t)$, find α, β, A, and B.
   We have

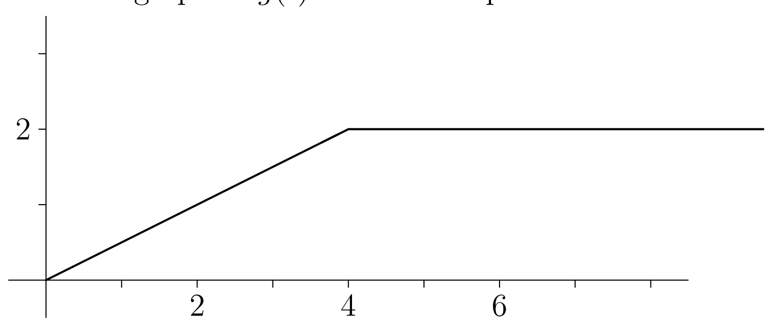
$$\begin{aligned} y(t) &= 3(\cos(22t) + \cos(24t)) + 2\cos(24t) \\ &= 3(2\cos(t)\cos(23t)) + 2\cos(24t) \\ &= 6\cos(t)\cos(23t) + 2\cos(24t). \end{aligned}$$

   (b) Assuming t is measured in seconds, how far apart are the amplitude peaks in $y(t)$?
   The amplitude peaks occur at the maxima of $|6\cos(t)|$, which are π seconds apart.

2. Let

$$g(t) = \begin{cases} t/2 & \text{if } 0 \le t < 4 \\ 2 & \text{if } t \ge 4 \end{cases}$$

   (a) Sketch a graph of $g(t)$ on the axes provided.



   (b) Find $\mathcal{L}\{g(t)\}$.

We first write $g(t)$ using step functions. We have

$$\begin{aligned} g(t) &= (1-u_4(t))\frac{t}{2} + 2u_4(t) \\ &= \frac{t}{2} + u_4(t)\left(-\frac{t}{2}+2\right) \\ &= \frac{t}{2} - \frac{1}{2}u_4(t)(t-4). \end{aligned}$$

Thus

$$\mathcal{L}\{g(t)\} = \frac{1}{2s^2} - \frac{e^{-4s}}{2s^2}, \quad s > 0.$$

3. Let $g(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 4 \\ 1 & \text{if} \quad 4 \le t < 6 \\ 0 & \text{if} \quad t \ge 6. \end{cases}$

   Use the *definition of the Laplace transform* (not the table) to compute $\mathcal{L}\{g(t)\}$.

   By definition,

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\, dt \\ &= \int_4^6 e^{-st}\, dt \\ &= -\frac{1}{s}\left[e^{-st}\right]_{t=4}^{t=6} \\ &= -\frac{1}{s}(e^{-6s} - e^{-4s}) \\ &= \frac{e^{-4s} - e^{-6s}}{s}. \end{aligned}$$

4. For each given $Y(s)$, find $\mathcal{L}^{-1}\{Y(s)\}$.

   (a) $Y(s) = \dfrac{s+5}{s^2 - 3s - 10}$.

   The denominator factors as $(s-5)(s+2)$, so we use partial fractions.

We have

$$\frac{A}{s-5} + \frac{B}{s+2} \quad = \quad \frac{s+5}{s^2-3s-10},$$

from which we get $A = \dfrac{10}{7}$ and $B = -\dfrac{3}{7}$. Thus

$$Y(s) \quad = \quad \frac{10}{7}\frac{1}{s-5} - \frac{3}{7}\frac{1}{s+2}$$

and so

$$\mathcal{L}^{-1}\{Y(s)\} \quad = \quad \frac{10}{7}e^{5t} - \frac{3}{7}e^{-2t}.$$

(b) $Y(s) = \dfrac{3s+2}{s^2+4s+10}$.

We complete the square in the denominator, getting $(s+2)^2+6$. We rewrite the numerator in terms of $s+2$, getting $3(s+2)-4$. We have

$$Y(s) \quad = \quad \frac{3(s+2)}{(s+2)^2+6} - \frac{4}{\sqrt{6}}\frac{\sqrt{6}}{(s+2)^2+6}$$

so that

$$y(t) \quad = \quad e^{-2t}\left(3\cos(\sqrt{6}t) - \frac{4}{\sqrt{6}}\sin(\sqrt{6}t)\right).$$

(c) $Y(s) = \dfrac{1-3e^{-2s}}{s(s^2+9)}$

We first apply partial fractions to get

$$\frac{1}{s(s^2+9)} \quad = \quad \frac{1}{9}\frac{1}{s} - \frac{1}{9}\frac{s}{s^2+9}.$$

Writing $H(s) = \dfrac{1}{9}\dfrac{1}{s} - \dfrac{1}{9}\dfrac{s}{s^2+9}$, we let $h(t) = \mathcal{L}^{-1}\{H(s)\}$, so that

$$h(t) \quad = \quad \frac{1}{9} - \frac{1}{9}\cos(3t).$$

Then

$$\begin{aligned} y(t) &= h(t) - 3u_2(t)h(t-2) \\ &= \frac{1}{9}(1-\cos 3t) - \frac{1}{3}u_2(t)(1-\cos(3(t-2))). \end{aligned}$$

5. Consider the initial value problem $y'' + 2y' + 10y = \delta(t-2); \quad y(0) = 3, \quad y'(0) = -1.$ Find $Y(s)$, the Laplace transform of the solution. You do not need to find the solution.

   The Laplace transform of the left side is

$$s^2Y(s) - 3s + 1 + 2(sY(s) - 3) + 10Y(s) \quad = \quad (s^2 + 2s + 10)Y(s) - 3s - 5.$$

   The Laplace transform of the right side is simply e^{-2s}. Solving the equation

$$(s^2 + 2s + 10)Y(s) - 3s - 5 \quad = \quad e^{-2s}$$

   for $Y(s)$, we get

$$Y(s) \quad = \quad \frac{3s + 5 + e^{-2s}}{s^2 + 2s + 10}.$$