Math 333 | Solution
Final Exam | May 2003

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1. Solve the first-order initial value problem $y' - 2ty = \dfrac{e^{t^2}}{t}$; $\quad y(1) = 2$. (Assume $t > 0$.)

   Solution: The integrating factor is e^{-t^2}. Multiplying through by e^{-t^2}, we get

   $$e^{-t^2}y' - 2te^{-t^2}y = \frac{1}{t}$$

   so that

   $$\begin{aligned} e^{-t^2}y &= \ln t + C \\ y &= e^{t^2}\ln t + Ce^{t^2} \end{aligned}$$

   Given that $y(1) = 2$, we have

   $$2 = Ce$$

   so that $C = 2e^{-1}$, and

   $$y = e^{t^2}\ln t + 2e^{t^2-1}.$$

2. Solve the first-order initial value problem $y' + y^2\cos x = 0$; $\quad y(0) = 4$.

   Solution: This equation is separable. We get

   $$\frac{y'}{y^2} = -\cos x$$

   and integrate both sides to get

   $$\begin{aligned} -\frac{1}{y} &= -\sin x + c_1 \\ y &= \frac{1}{\sin x + c_2}. \end{aligned}$$

   Now we impose the initial condition, getting

   $$4 = \frac{1}{0 + c_2}$$

whence $c_2 = \frac{1}{4}$. The solution is

$$y = \frac{1}{\sin x + (1/4)}.$$

3. A force of 10 pounds stretches a spring 4 inches. The spring's internal friction exerts a viscous damping force of 3 pound-seconds per inch. A 5-pound weight is attached to the spring, pushed upward 6 inches from its rest position, and then released at $t = 0$.

   Write an initial value problem for $u(t)$, the position of the mass at time t. Be sure to specify units for $u(t)$. You do not need to solve the initial value problem.

   (Use 32 ft/sec^2 for gravitational acceleration.)

   Solution: The spring constant is 10 pounds per 4 inches, or 30 pounds per foot, so we have $k = 30$.

   To find m, we note that $32m = 5$, so $m = \frac{5}{32}$ lb-sec^2 per foot.

   The damping constant γ is given as 3 pound-seconds per inch, which is 36 pound-seconds per foot.

   Here's the initial value problem:

$$\frac{5}{32}u'' + 36u' + 30u = 0; \quad u(0) = -\frac{1}{2}; \quad u'(0) = 0$$

   with $u(t)$ in feet.

   We could also solve the problem with $u(t)$ in inches. In this case, the spring constant is $\frac{5}{2}$ pounds per inch, so we have $k = \frac{5}{2}$ pounds per inch.

   Gravitational acceleration is $32 \times 12 = 384$ in/sec^2, so we have $384m = 5$, and

$$m = \frac{5}{384}\frac{\text{lb-sec}^2}{\text{in}}.$$

   The damping constant γ is already given in the correct units: 3 pound-seconds per inch. We have

$$\frac{5}{384}u'' + 3u' + \frac{5}{2}u = 0; \quad u(0) = -6; \quad u'(0) = 0$$

   with $u(t)$ in inches.

4. (a) Find the equilibrium solutions to the differential equation $y' = y^3 - 3y^2 - 4y$, and classify each as asymptotically stable, unstable, or semi-stable.

Solution: We can factor $y^3 - 3y^2 - 4y$ as $y(y-4)(y+1)$. The equilibrium solutions are

$y = -1$, unstable
$y = 0$, asymptotically stable
$y = 4$, unstable.

(b) Suppose $y_c(t)$ is a solution to the initial value problem $y' = y^3 - 3y^2 - 4y$; $y(0) = c$. Find $\lim_{t\to\infty} y_c(t)$ if

i. $c = -2$
Solution: This solution goes to $-\infty$ as $t \to \infty$.

ii. $c = -1$
Solution: This solution is an equilibrium; $y_{-1}(t) = -1$ for all t, so $\lim_{t\to\infty} y_{-1}(t) = -1$.

iii. $c = 3$
Solution: As $t \to \infty$, this solution approaches the asymptotically stable solution $y = 0$.

5. The function $y_1(t) = t^2$ is one solution to the differential equation

$$t^2y'' - 3ty' + 4y = 0, \quad t > 0$$

Suppose $y_2(t) = t^2v(t)$ is a second solution. Find the most general form for $v(t)$.

Solution: If $y_2 = t^2v$, then $y_2' = 2tv + t^2v'$ and $y_2'' = 2v + 4tv' + t^2v''$, so that we get

$$\begin{aligned} 0 &= t^2(2v + 4tv' + t^2v'') - 3t(2tv + t^2v') + 4t^2v \\ &= t^4v'' + t^3v' \\ &= t^3(tv'' + v') \end{aligned}$$

Since $t > 0$, we can divide through by t and then make the subsitution $u = v'$ to get

$$tu' + u = 0$$

This is separable; we write $\frac{u'}{u} = -\frac{1}{t}$ so that $\ln u = -\ln t + c_1$, and so

$$\begin{aligned} u &= e^{-\ln t + c_1} \\ &= \frac{c_2}{t}. \end{aligned}$$

We then integrate to get

$$v(t) = c_2 \ln t + c_3.$$

6. Compute $\mathcal{L}^{-1}\left\{\dfrac{7s+10}{s(s^2+4s+5)}\right\}$

Solution: We write

$$\frac{7s+10}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

and solve for A, B, and C to get

$$\begin{aligned}
\frac{7s+10}{s(s^2+4s+5)} &= \frac{2}{s} - \frac{2s+1}{s^2+4s+5} \\
&= \frac{2}{s} - \frac{2s+4}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} \\
&= \frac{2}{s} - \frac{2(s+2)}{(s+2)^2+1} + \frac{3}{(s+2)^2+1}
\end{aligned}$$

We find the inverses of these terms in the table. We get

$$\mathcal{L}^{-1}\left\{\frac{7s+10}{s(s^2+4s+5)}\right\} = 2 - 2e^{-2t}\cos t + 3e^{-2t}\sin t.$$

7. Consider the initial value problem $y'' + 5y' + 4y = g(t) + \delta(t-2)$, $y(0) = 2$, $y'(0) = 0$, where

$$g(t) = \begin{cases} 0 & \text{if} \quad t < 2 \\ 3t - 6 & \text{if} \quad t \geq 2 \end{cases}$$

Find $Y(s)$, the Laplace transform of the solution. You do not need to find the solution itself.

Solution: First, we write $g(t) = u_2(t)(3(t-2))$, so that $\mathcal{L}\{g(t)\} = \dfrac{3e^{-2s}}{s^2}$.

Next we take the Laplace transforms of both sides of the given equation, getting

$$\begin{aligned}
s^2Y - 2s + 5(sY - 2) + 4Y &= \frac{3e^{-2s}}{s^2} + e^{-2s} \\
(s^2 + 5s + 4)Y &= 2s + 10 + e^{-2s}\left(\frac{3}{s^2} + 1\right) \\
Y(s) &= \frac{1}{s^2 + 5s + 4}\left(2s + 10 + e^{-2s}\left(\frac{3}{s^2} + 1\right)\right).
\end{aligned}$$

8. Suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution to the differential equation $y'' + 2xy' + 2y = 0$.

   Find the recurrence relation for the coefficients a_n. (You should get a formula giving a_{n+2} in terms of a_n.)

   Solution: With y as given, we have

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

   so that

$$2xy' = \sum_{n=1}^{\infty} 2n a_n x^n.$$

   Differentiating again, we get

$$\begin{aligned}
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\
&= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n
\end{aligned}$$

   The differential equation reads

$$\sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+2} + 2na_n + 2a_n)x^n = 0.$$

   We have the recurrence relation

$$(n+1)(n+2)a_{n+2} + (2n+2)a_n = 0$$

which we simplify to

$$a_{n+2} = -\frac{2a_n}{n+2}.$$

Bonus: Let $\sum_{n=0}^{\infty} a_n x^n$ be the power series solution to the initial value problem $y' = 1 + y^2$; $y(0) = 0$.

Find a_0, a_1, a_2, a_3, a_4, and a_5

Solution: If $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and $y(0) = a_0 = 0$, then we get

$$\begin{aligned} 1 + y^2 &= 1 + (a_1x + a_2x^2 + a_3x^3 + \cdots)(a_1x + a_2x^2 + a_3x^3 + \cdots) \\ &= 1 + a_1^2x^2 + 2(a_1a_2)x^3 + (2a_1a_3 + a_2^2)x^4 + (2a_1a_4 + 2a_2a_3)x^5 + \cdots \end{aligned}$$

We also have

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$$

Equating the coefficients in the two power series yields:

$$\begin{aligned} a_1 &= 1 \\ 2a_2 &= 0 \\ 3a_3 &= a_1^2 \\ 4a_4 &= 2a_1a_2 \\ 5a_5 &= 2a_1a_3 + a_2^2 \end{aligned}$$

Putting this all together, we get

$$a_0 = 0; \quad a_1 = 1; \quad a_2 = 0; \quad a_3 = \frac{1}{3}; \quad a_4 = 0; \quad a_5 = \frac{1}{15}.$$