In this example, we'll prove an assertion on page 117 of Boyce & DiPrima about solutions of a linear difference equation. Our notation will differ slightly from the notation in the book, so that we can untangle the solution of a difference equation from the difference equation itself.

Claim: Let ρ and b be constants, with $\rho \neq 1$. Suppose the sequence $\phi_0, \phi_1, \phi_2, \ldots$ is a solution to the difference equation

$$y_{n+1} = \rho y_n + b \tag{1}$$

with initial value $y_0 = \alpha$.

Then for each integer $n \geq 1$ we have

$$\phi_n = \rho^n \alpha + \left(\frac{1-\rho^n}{1-\rho}\right) b.$$

Comment: What we're trying to prove is actually an infinite sequence of statements, which we might call P_1, P_2, P_3, ... Statement P_1 is that $\phi_1 = \rho^1\alpha + \left(\frac{1-\rho^1}{1-\rho}\right) b$, statement P_2 is that $\phi_2 = \rho^2\alpha + \left(\frac{1-\rho^2}{1-\rho}\right) b$, and so on. So mathematical induction is the right method to use. We'll verify that P_1 is true, and then show that whenever P_k is true, P_{k+1} is also true.

Proof: We proceed by induction.

<u>Base case: $n = 1$</u>.

We need to verify that $\phi_1 = \rho\alpha + \left(\frac{1-\rho}{1-\rho}\right) b$. Since $\rho \neq 0$, we know that

$$\frac{1-\rho}{1-\rho} = 1,$$

so what we're trying to show is that

$$\phi_1 = \rho\alpha + b. \tag{2}$$

We are given that the sequence $\phi_0, \phi_1, \phi_2, \ldots$ is a solution to the difference equation (1) with given initial condition, so we know that $\phi_0 = \alpha$ and that

$$\begin{aligned} \phi_1 &= \rho\phi_0 + b \\ &= \rho\alpha + b. \end{aligned}$$

This shows that equation (2) does indeed hold, and the base case is complete.

Comment: We've shown that the statement P_1 is true.

Inductive step.

Now let k be an integer with $k \geq 1$, and suppose that equation (2) holds when $n = k$. That is, suppose that

$$\phi_k = \rho^k \alpha + \left(\frac{1-\rho^k}{1-\rho}\right) b. \tag{3}$$

We will show that equation (2) also holds when $n = k + 1$.

Comment: That is, we'll show that whenever statement P_k is true, statement P_{k+1} is also true.

We are given that the sequence $\phi_0, \phi_1, \phi_2, \ldots$ is a solution to the difference equation (1), and that k is at least 1, so we know that

$$\phi_{k+1} = \rho \phi_k + b.$$

We use our assumption in line (3) above to make a substitution for ϕ_k, getting

$$\phi_{k+1} = \rho \left(\rho^k \alpha + \left(\frac{1-\rho^k}{1-\rho}\right) b\right) + b.$$

Now we distribute the ρ and do a little factoring to get

$$\phi_{k+1} = \rho^{k+1} \alpha + \left(\frac{\rho(1-\rho^k)}{1-\rho} + 1\right) b. \tag{4}$$

The expression in parentheses is

$$\frac{(\rho - \rho^{k+1}) + (1-\rho)}{1-\rho} = \frac{1-\rho^{k+1}}{1-\rho},$$

so equation (4) reads

$$\phi_{k+1} = \rho^{k+1} \alpha + \left(\frac{1-\rho^{k+1}}{1-\rho}\right) b,$$

which is exactly equation (2) with $n = k + 1$. This completes the inductive step, and the proof.