A1. For each operator, either prove that the operator is linear or find an example showing that it is not linear.

(a) $F_1[y] = 2y'' + y$.
Solution: This operator is linear. For functions y_1 and y_2 and a constant c, we have

$$\begin{aligned} F_1[cy_1] &= 2(cy_1)'' + cy_1 \\ &= 2cy_1'' + cy_1 \\ &= c(2y_1'' + y_1) \\ &= cF[y_1], \end{aligned}$$

and

$$\begin{aligned} F_1[y_1 + y_2] &= 2(y_1 + y_2)'' + (y_1 + y_2) \\ &= 2(y_1'' + y_2'') + (y_1 + y_2) \\ &= 2y_1'' + 2y_2'' + y_1 + y_2 \\ &= 2y_1'' + y_1 + 2y_2'' + y_2 \\ &= F_1[y_1] + F_1[y_2]. \end{aligned}$$

(b) $F_2[y] = e^y$.
Solution: This is not linear. Let y be the constant function $y(t) = 0$ and $c = 2$. Then

$$\begin{aligned} F_2[cy] &= F[2 \cdot 0] \\ &= F[0] \\ &= e^0 \\ &= 1, \end{aligned}$$

while

$$\begin{aligned} cF_2[y] &= 2e^y \\ &= 2e^0 \\ &= 2. \end{aligned}$$

Since $1 \neq 2$, the operator is not linear.

(c) $F_3[y] = y^2 + y'$.

Solution: This operator is not linear. Let y be the constant function $y(t) = 2$. Let $c = 2$. We have

$$\begin{aligned} F[cy] &= F[2 \cdot 2] \\ &= F[4] \\ &= 16 + 0 \quad \text{since the derivative of a constant is 0} \\ &= 16, \end{aligned}$$

while

$$\begin{aligned} 2F[y] &= 2 \cdot (2^2 + 0) \\ &= 8. \end{aligned}$$

Since $16 \neq 8$, the operator is not linear.

A2. Verify that $y = te^{-2t} + t - 1$ is a solution to $y'' + 4y' + 4y = 4t$.

Solution: We have

$$\begin{aligned} y(t) &= te^{-2t} + t - 1, \\ y'(t) &= -2te^{-2t} + e^{-2t} + 1 \\ y''(t) &= 4te^{-2t} - 4e^{-2t}. \end{aligned}$$

Plugging these expressions into the left side of the given differential equation yields

$$\begin{aligned} y''(t) + 4y'(t) + 4y(t) &= (4te^{-2t} - 4e^{-2t}) + 4(-2te^{-2t} + e^{-2t} + 1) + 4(te^{-2t} + t - 1) \\ &= (4 - 8 + 4)e^{-2t} + (-4 + 4)e^{-2t} + 4t + (4 - 4) \\ &= 4t. \end{aligned}$$

B1. For what values of α is $y(t) = \cos(\alpha t)$ a solution to $y'' + 5y = 0$?

Solution: Setting $y(t) = \cos(\alpha t)$, we find that $y''(t) = -\alpha^2 \cos(\alpha t)$, so that

$$\begin{aligned} y'' + 5y &= -\alpha^2 \cos(\alpha t) + 5\cos(\alpha t) \\ &= (5 - \alpha^2)\cos(\alpha t). \end{aligned}$$

If $\alpha = \pm\sqrt{5}$, then the expression above is zero for all values of t. If, on the other hand, α is any other value, then $(5 - \alpha^2) \neq 0$, and so at $t = 0$, the expression $(5 - \alpha^2)\cos(\alpha t)$ is non-zero. Thus we must have $\alpha = \pm\sqrt{5}$ for $\cos(\alpha t)$ to be a solution to the given differential equation.

B2. For what values of r is $y(t) = e^{rt}$ a solution to $y'' + 3y' - 10y = 0$?

Solution: Setting $y(t) = e^{rt}$, we find that $y'(t) = re^{rt}$ and $y''(t) = r^2 e^{rt}$, so that

$$\begin{aligned} y'' + 3y' - 10y &= r^2 e^{rt} + 3re^{rt} - 10e^{rt} \\ &= (r^2 + 3r - 10)e^{rt}. \end{aligned}$$

Since e^{rt} is not zero, we need $r^2 + 3r - 10 = 0$, and this happens when $r = 2$ or $r = -5$.

B3. For what values of r is $y(t) = t^r$ a solution to $t^2 y'' + 3ty' - 8y = 0$?

Solution: Setting $y(t) = t^r$, we find that $y'(t) = rt^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$, so that

$$\begin{aligned} t^2 y'' + 3ty' - 8y &= t^2 r(r-1)t^{r-2} + 3trt^{r-1} - 8t^r \\ &= r(r-1)t^r + 3rt^r - 8t^r \\ &= (r^2 + 2r - 8)t^r \\ &= (r+4)(r-2)t^r. \end{aligned}$$

Now the function t^r is non-zero whenever $t \neq 0$, so so for the expression above to be the zero function, we must have $r = -4$ or $r = 2$.

C1. By hand, draw direction fields for the following first-order differential equations.

(a) $\dfrac{dy}{dt} = y - \dfrac{t}{2}$

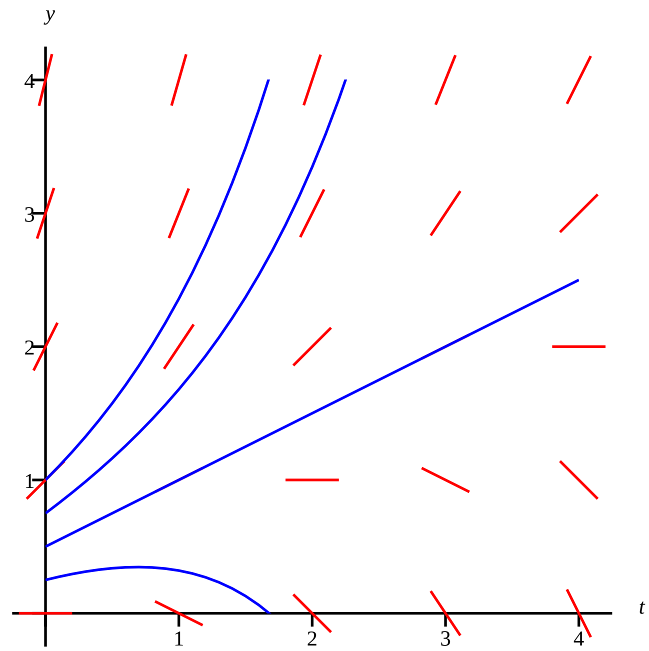
(b) $\dfrac{dy}{dt} = \dfrac{(t-2)^2}{2} - y$

Use as grid points the integer lattice points in the ty-plane with $0 \leq t \leq 4$ and $0 \leq y \leq 4$.

Sketch a few solution curves on each direction field. Try to predict the long-term behavior of solutions to these differential equations.

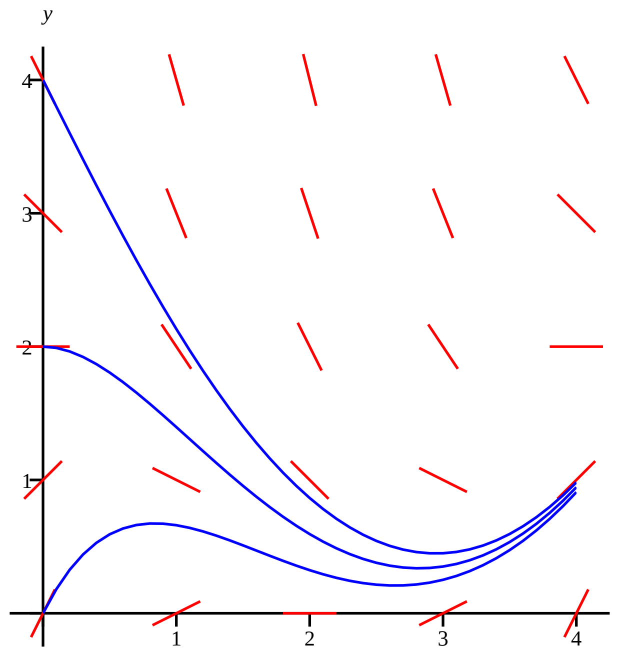
Solution:

(a) Here's a drawing showing the direction field and a few solution curves.



Some solutions increase exponentially and some decrease exponentially. The dividing line seems to be at $y = (t+1)/2$.

(b) Here's a drawing showing the direction field and a few solution curves.



This time, it looks as though there's a parabola P that divides the decreasing solutions from the increasing solutions. The solutions starting above the P decrease to P; those staring below P increase to P.

C2. Use Maple to draw direction fields for the following first-order differential equations.

(a) $\dfrac{dy}{dt} = y^3 - 2y$

(b) $\dfrac{dy}{dt} = \cos(\pi t) - y$

Hand in printouts of each Maple plot. Sketch a few solution curves on each one. Use the region $0 \le t \le 6$ and $-2 \le y \le 2$ in the ty-plane. Try to predict the long-term behavior of solutions to these differential equations.

Here's what you need to know about Maple. Start by loading the differential equations package with the command

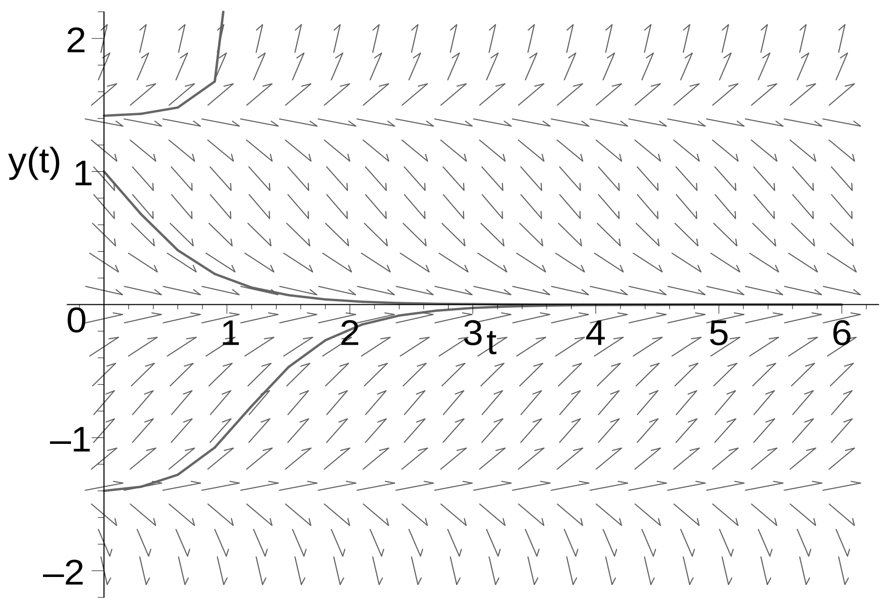
```
> with(DEtools):
```

To draw a direction field, use the command `dfieldplot`. Here's the syntax for part C2a:

```
> dfieldplot(diff(y(t),t)=y(t)^3-2*y(t),y(t),t=0..6,y=-2..2);
```

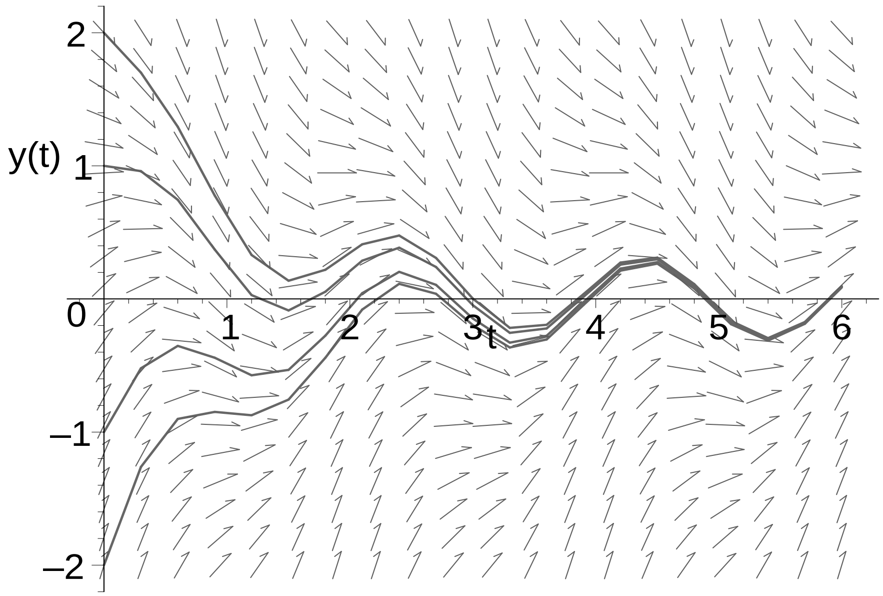
Solution:

(a) Here's the Maple plot with a few solution curves drawn in.



It looks like solutions that start between about -1.4 and 1.4 all approach zero, and solutions that start outside this range go quickly to plus or minus infinity.

(b) Here's the Maple plot with a few solution curves drawn in.



It looks as though all solution curves approach a sinusoid.