Math 333 Problem Set #2 Due 14 February 2003, 12:00 NOON

Reading: B & D, §§2.1 – 2.3

Exercises: Write your solutions clearly, and in complete sentences, remembering that they will be graded for presentation as well as correctness. Please prepare **separate** solution sets to the *A* problems, the *B* problems, and the *C* problems. You will hand them in in different places.

A1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + e^{3x}}{2y - 4}; \quad y(0) = 1.$$

A2. Consider the initial value problem $y' + (\cot t)y = 2\sin t; \quad y(\pi/2) = y_0$.

(a) Use the method of integrating factors to solve the initial value problem. (Your solution will depend on y_0.)

(b) Use the computer to draw a direction field for this differential equation with $0 \le t \le \pi$ and $-3 \le y \le 3$.

(c) As you can tell from the direction field, for some values of y_0 (that is, $y(\pi/2)$), solutions to this IVP go to $+\infty$ as t approaches π, and for other values of y_0, the solutions go to $-\infty$ as t approaches π. There's a critical value y_c, such that solutions with $y_0 < y_c$ go to $-\infty$ and solutions with $y_0 > y_c$ go to $+\infty$. Determine the value of y_c.

B1. (B & D, Section 2.3, problem 14) When an organism dies, the amount of carbon-14 it contains begins to decay at a rate proportional to the amount present. After about 5730 years, only half of the organism's original carbon-14 is left. Let $Q(t)$ denote the amount of carbon-14 in some organism's remains.

(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14. Give the value of r in its exact form (using logarithms) and then give a decimal approximation.

(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

B2. (B & D, Section 2.3, problem 3) A tank originally contains 100 gal of fresh water. Then water containing 1/2 lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

C1. (Based on B & D, Section 2.3, problem 10) Consider a mortgage in which an initial amount B_0 is borrowed, interest is charged at a rate r (compounded continuously), and payments are made at a rate k, assumed continuous. Let $B(t)$ denote the amount of money still owed at time t.

(a) Construct a model (using appropriate units) and solve the initial value problem.

(b) (From problem 10) A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of a mortgage is 20 years. What is the maximum amount the buyer can afford to borrow, and how much interest will be paid on this amount through the term of the mortgage?

C2. A tank contains 100 liters of fresh water. A salt solution flows into the tank at the rate of 10 liters per minute, and the mixture in the tank flows out at the same rate. The concentration of salt in the incoming solution is given by $50 + 10\sin(t)$ grams per liter at time t (in minutes).

(a) Write an initial value problem for $Q(t)$, the quantity of salt in the tank at time t.

(b) Solve the IVP. You may use a computer to do some of the nasty integration, but don't rely on `dsolve()` or anything like that; that would take all the fun out of it.

(c) Use the computer to draw a graph showing the *concentration* of salt in the tank along with the concentration of salt in the incoming solution. Take t to at least 60 minutes. How would you describe the long-term behavior of the concentration of salt in the tank?