A1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2 + e^{3x}}{2y - 4}; \quad y(0) = 1.$$

Solution: This is separable; we write

$$2y - 4\,dy = x^2 + e^x\,dx$$

and integrate to get

$$y^2 - 4y = \frac{x^3}{3} + \frac{e^{3x}}{3} + C.$$

The initial condition says

$$-3 = \frac{1}{3} + C,$$

so $C = -\dfrac{10}{3}$, and we have

$$y^2 - 4y = \frac{x^3 + e^{3x} - 10}{3}.$$

To solve this for y, we complete the square. We get

$$y^2 - 4y + 4 = \frac{x^3 + e^{3x} + 2}{3}$$

so that

$$(y - 2) = \pm\sqrt{\frac{x^3 + e^{3x} + 2}{3}}.$$

The initial condition says that we have to take the negative square root, so the solution is

$$y = 2 - \sqrt{\frac{x^3 + e^{3x} + 2}{3}}.$$

A2. Consider the initial value problem $y' + (\cot t)y = 2\sin t; \quad y(\pi/2) = y_0$.

(a) Use the method of integrating factors to solve the initial value problem. (Your solution will depend on y_0.)

Solution: The integrating factor here is $e^{\int \cot t\, dt}$, which turns out to be simply $\sin t$. Multiplying through by $\sin t$, we get

$$(\sin t)y' + (\cos t)y = 2\sin^2 t.$$

Integrating both sides, we get

$$\begin{aligned}(\sin t)y &= t - \frac{\sin 2t}{2} + C \\ &= t - \cos t \sin t + C.\end{aligned}$$

We solve for y, getting

$$y = \frac{t + C}{\sin t} - \cos t.$$

The initial condition says that

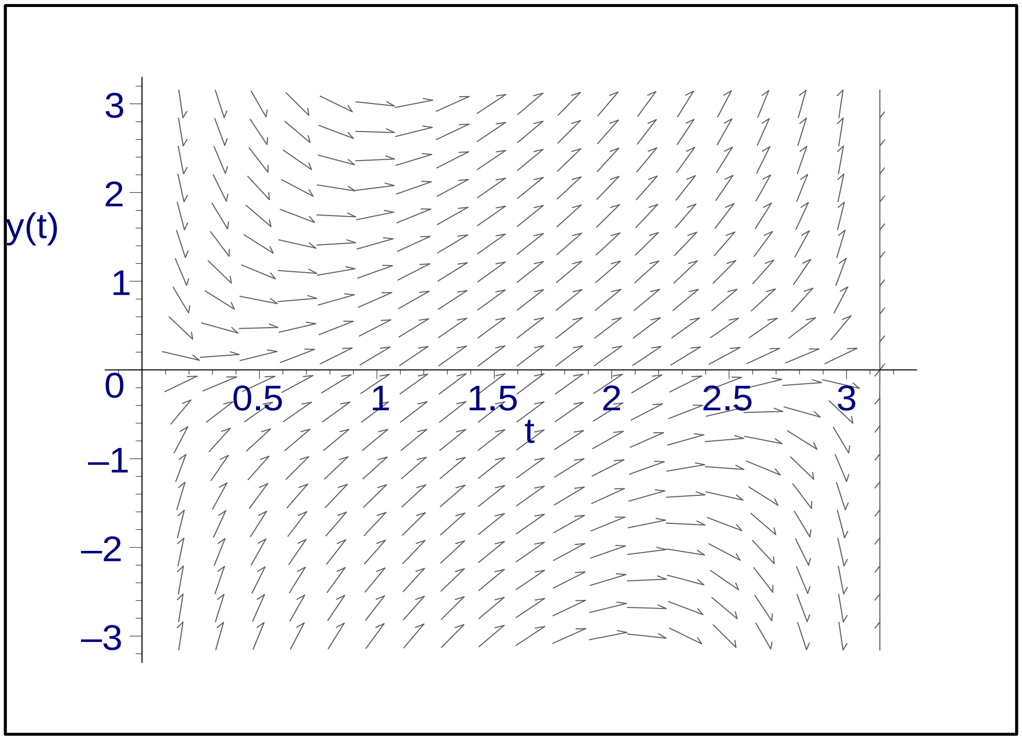
$$y_0 = \frac{\pi}{2} + C$$

so that $C = y_0 - \dfrac{\pi}{2}$, and our solution is

$$y = \frac{t + y_0 - \frac{\pi}{2}}{\sin t} - \cos t.$$

(b) Use the computer to draw a direction field for this differential equation with $0 \le t \le \pi$ and $-3 \le y \le 3$.

Solution: Here is the picture.



(c) As you can tell from the direction field, for some values of y_0 (that is, $y(\pi/2)$), solutions to this IVP go to $+\infty$ as t approaches π, and for other values of y_0, the solutions go to $-\infty$ as t approaches π. There's a critical value y_c, such that solutions with $y_0 < y_c$ go to $-\infty$ and solutions with $y_0 > y_c$ go to $+\infty$. Determine the value of y_c.

Solution: The solutions go to $\pm\infty$ as $t \to \pi^-$ because the denominator $\sin t$ goes to zero there (from above). A solution will go to $+\infty$ if the numerator, $t + y_0 - \dfrac{\pi}{2}$, is positive as $t \to \pi^-$. This happens when

$$\begin{aligned} \pi + y_0 - \frac{\pi}{2} &> 0 \\ \frac{\pi}{2} + y_0 &> 0, \end{aligned}$$

that is, when $y_0 > -\dfrac{\pi}{2}$. Similarly, when $y_0 < -\dfrac{\pi}{2}$, the expression $t + y_0 - \dfrac{\pi}{2}$ is negative when $t \to \pi^-$, so the solution runs off to $-\infty$.

B1. (B & D, Section 2.3, problem 14) When an organism dies, the amount of carbon-14 it contains begins to decay at a rate proportional to the amount present. After about 5730 years, only half of the organism's original carbon-14 is left. Let $Q(t)$ denote the amount of carbon-14 in some organism's remains.

(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14. Give the value of r in its exact form (using logarithms) and then give a decimal approximation.

Solution: The solution to the differential equation is

$$Q(t) = Q_0 e^{-rt}.$$

The given condition says that

$$\frac{Q_0}{2} = Q_0 e^{-5730r}.$$

We solve this for r, getting

$$\begin{aligned} r &= \frac{\ln 2}{5730} \\ &\approx 1.20968 \times 10^{-4}. \end{aligned}$$

(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

Solution: We've already done this. We have

$$Q(t) = Q_0 e^{-(\ln(2)/5730)t}.$$

(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

Solution: The condition given says that

$$\frac{Q_0}{5} = Q_0 e^{-(\ln(2)/5730)t}$$

We cancel the Q_0's and take logarithms to get

$$\begin{aligned} -\ln 5 &= -\left(\frac{\ln 2}{5730}\right) t \\ t &= \frac{5730 \ln 5}{\ln 2} \\ &\approx 13,304.65 \text{ years.} \end{aligned}$$

B2. (B & D, Section 2.3, problem 3) A tank originally contains 100 gal of fresh water. Then water containing 1/2 lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

Solution: Let $Q(t)$ denote the amount of salt in the tank (in pounds) at time t (in minutes). For the first ten minutes of the experiment, $Q(t)$ satisfies the differential equation

$$Q'(t) = 2 \times \frac{1}{2} - 2 \times \frac{Q(t)}{100}$$

with the initial condition $Q(0) = 0$. We can solve this differential equation using integrating factors. We write

$$Q'(t) + \frac{Q(t)}{50} = 1$$

and multiply through by $e^{\frac{t}{50}}$ to get

$$\begin{aligned} e^{\frac{t}{50}} Q'(t) + e^{\frac{t}{50}} \frac{Q(t)}{50} &= e^{\frac{t}{50}} \\ \frac{d}{dt}\left(e^{\frac{t}{50}} Q(t)\right) &= e^{\frac{t}{50}} \end{aligned}$$

and integrate to get

$$e^{\frac{t}{50}} Q(t) = 50 e^{\frac{t}{50}} + C.$$

The initial condition implies that $C = -50$. We divide through by $e^{\frac{t}{50}}$ to get

$$\begin{aligned} Q(t) &= 50 - 50e^{-\frac{t}{50}} \\ &= 50\left(1 - e^{-\frac{t}{50}}\right). \end{aligned}$$

After ten minutes, the amount of salt in the tank is

$$Q(10) = 50(1 - e^{-\frac{1}{5}}).$$

For the next ten minutes, the function $Q(t)$ satisfies the differential equation

$$\begin{aligned} Q'(t) &= -2 \times \frac{Q(t)}{100} \\ &= -\frac{Q(t)}{50}. \end{aligned}$$

This one we can solve by inspection; the solution is

$$Q(t) = Q_0 e^{-\frac{t}{50}},$$

where Q_0 is the amount of salt in the tank at the beginning of the second ten minutes. (We re-set the clock after the first ten minutes.) We get this amount from above; it's $50(1 - e^{-\frac{1}{5}})$. Thus after another ten minutes, the amount of salt in the tank is

$$\begin{aligned} Q(10) &= 50(1 - e^{-\frac{1}{5}})e^{-\frac{1}{5}} \\ &= 50(e^{-\frac{1}{5}} - e^{-\frac{2}{5}}) \\ &\approx 7.42 \text{ pounds.} \end{aligned}$$

Alternatively, we could start the second differential equation running on the same clock as the first, in which case we have

$$Q(t) = Ce^{-\frac{t}{50}}$$

with the initial condition $Q(10) = 50(1 - e^{-\frac{1}{5}})$. Using this initial condition to determine C, we get

$$50(1 - e^{-\frac{1}{5}}) = Ce^{-\frac{1}{5}}$$

so that $C = 50(e^{\frac{1}{5}} - 1)$. We have

$$Q(t) = 50(e^{\frac{1}{5}} - 1)e^{-\frac{t}{50}}$$

where t is the time since the very beginning of the experiment. The answer to the question should now be

$$\begin{aligned} Q(20) &= 50(e^{\frac{1}{5}} - 1)e^{-\frac{2}{5}} \\ &= 50(e^{-\frac{1}{5}} - e^{-\frac{2}{5}}) \\ &\approx 7.42 \text{ pounds.} \end{aligned}$$

C1. (Based on B & D, Section 2.3, problem 10) Consider a mortgage in which an initial amount B_0 is borrowed, interest is charged at a rate r (compounded continuously), and payments are made at a rate k, assumed continuous. Let $B(t)$ denote the amount of money still owed at time t.

(a) Construct a model (using appropriate units) and solve the initial value problem.

Solution: Taking r as a yearly percentage rate, k in dollars per year, $B(t)$ in dollars, and t in years, we have the initial value problem

$$B'(t) = rB(t) - k; \quad B(0) = B_0.$$

The differential equation is not difficult to solve. We write

$$B'(t) - rB(t) = -k$$

and multiply through by the integrating factor e^{-rt} to get

$$\frac{d}{dt}(e^{-rt}B) = -ke^{-rt}$$

so that

$$\begin{aligned} e^{-rt}B &= \frac{k}{r}e^{-rt} + C_1 \\ B &= \frac{k}{r} + C_1 e^{rt}. \end{aligned}$$

The initial condition says that

$$B_0 = \frac{k}{r} + C_1,$$

so we get $C_1 = B_0 - (k/r)$, and the solution to the IVP is

$$B(t) \;=\; \frac{k}{r} + \left(B_0 - \frac{k}{r}\right) e^{rt}.$$

(b) (From problem 10) A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of a mortgage is 20 years. What is the maximum amount the buyer can afford to borrow, and how much interest will be paid on this amount through the term of the mortgage?

Solution: The condition here is that $B(20) = 0$. That is,

$$0 \;=\; \frac{k}{r} + \left(B_0 - \frac{k}{r}\right) e^{20r}.$$

We solve this for B_0, getting

$$B_0 \;=\; \frac{k}{r}(1 - e^{-20r}).$$

The given values for k and r in the problem are $k = 9600$ dollars/year and $r = 0.09$. With these values, we get

$$\begin{aligned} B_0 &= \frac{9600}{0.09}(1 - e^{-1.8}) \\ &\approx \$89,034.78. \end{aligned}$$

Over the term of the mortgage, the borrower pays \$192,000. Since the principal is only \$89,034.78, the borrower ends up paying \$102,965.22 in interest.

C2. A tank contains 100 liters of fresh water. A salt solution flows into the tank at the rate of 10 liters per minute, and the mixture in the tank flows out at the same rate. The concentration of salt in the incoming solution is given by $50 + 10\sin(t)$ grams per liter at time t (in minutes).

(a) Write an initial value problem for $Q(t)$, the quantity of salt in the tank at time t.

Solution: We have

$$\begin{aligned} Q'(t) &= \text{(rate in)} - \text{(rate out)} \\ &= 10 \times (50 + 10\sin(t)) - 10 \times \frac{Q(t)}{100} \\ &= 500 + 100\sin t - \frac{10}{100} Q(t). \end{aligned}$$

The initial condition is $Q(0) = 0$.

(b) Solve the IVP. You may use a computer to do some of the nasty integration, but don't rely on `dsolve()` or anything like that; that would take all the fun out of it.

Solution: We write

$$Q'(t) + \frac{1}{10}Q(t) = 500 + 100\sin t$$

and use the integrating factor $e^{\frac{t}{10}}$ to get

$$\frac{d}{dt}e^{\frac{t}{10}}Q(t) = 500e^{\frac{t}{10}} + 100e^{\frac{t}{10}}\sin t.$$

Using the computer, we find that the integral of $e^{\frac{t}{10}}\sin t$ is

$$\frac{10}{101}e^{\frac{t}{10}}(\sin t - 10\cos t).$$

Thus when we integrate the equation above, we get

$$e^{\frac{t}{10}}Q(t) = 5000e^{\frac{t}{10}} + \frac{1000}{101}e^{\frac{t}{10}}(\sin t - 10\cos t) + C.$$

Solving for $Q(t)$, we get

$$Q(t) = 5000 + \frac{1000}{101}(\sin t - 10\cos t) + Ce^{-\frac{t}{10}}.$$

The initial condition says that

$$0 = 5000 + \frac{1000}{101}(-10) + C,$$

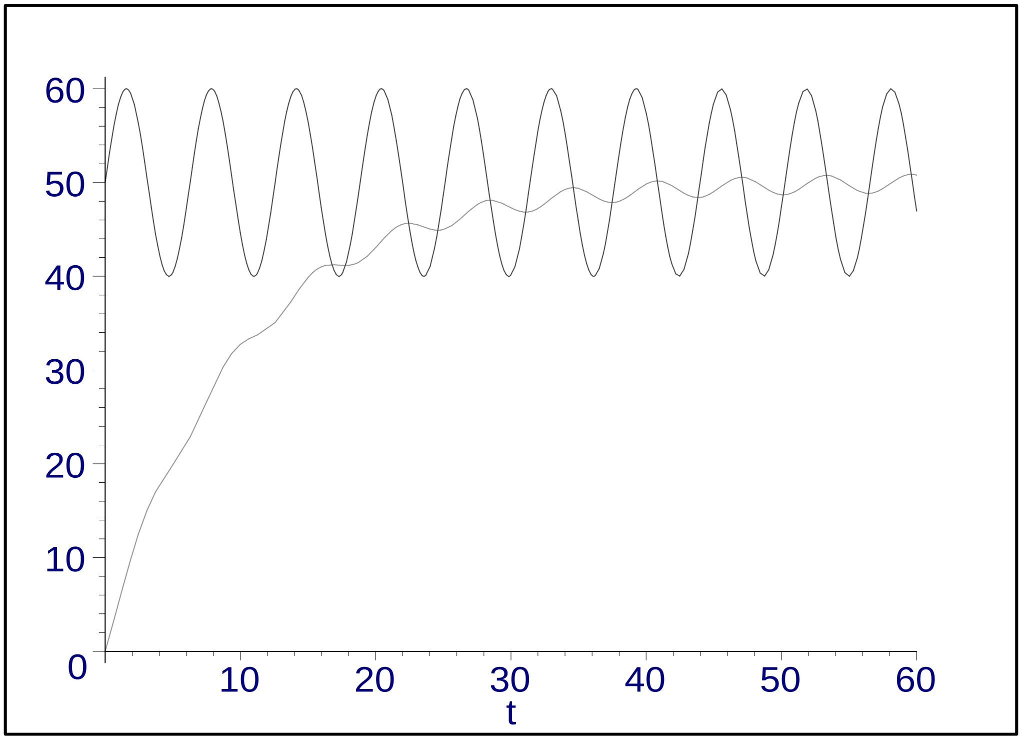
so $C = \dfrac{10000}{101} - 5000$.

Here, then, is the complete solution:

$$Q(t) = 5000 + \frac{1000}{101}(\sin t - 10\cos t) + \left(\frac{10000}{101} - 5000\right)e^{-\frac{t}{10}}.$$

(c) Use the computer to draw a graph showing the *concentration* of salt in the tank along with the concentration of salt in the incoming solution. Take t to at least 60 minutes. How would you describe the long-term behavior of the concentration of salt in the tank?

Solution: The concentration of salt in the tank is $Q(t)/100$, so we plot that function along with the function $50 + 10\sin(t)$. Here is the picture.



Here are some things to notice about the picture:

- The concentration of salt in the tank starts at zero and rises to a steady-state oscillation near 50 grams per liter. The rate at which the concentration rises varies directly with the concentration of the incoming solution.
- After about $t = 40$, the exponential term $Ce^{-t/10}$ becomes negligible. The steady-state solution is a sinusoid with the same period as the input function.
- The steady-state solution oscillates around 50 grams per liter, the same value around which the input concentration oscillates.
- The amplitude of the tank-concentration oscillation is smaller than the amplitude of the oscillation of the input solution. This makes sense; the volume

of water in the tank cushions the effect of the input.

- The concentration of salt in the tank increases whenever the salt concentration in the incoming solution is greater than the concentration in the tank, and decreases whenever the salt concentration in the incoming solution is less than the concentration in the tank.
- The oscillation of the salt concentration in the tank lags almost 90° behind the concentration salt in the incoming solution.