Math 333 Problem Set #3 Due 21 February 2003, 12:00 NOON

Reading: B & D, §2.5

Exercises: Write your solutions clearly, and in complete sentences, remembering that they will be graded for presentation as well as correctness. Please prepare **separate** solution sets to the A problems, the B problems, and the C problems. You will hand them in in different places.

A1. Consider the differential equation $\dfrac{dy}{dt} = y^3 - 3y^2 + 2y$. Plot $\dfrac{dy}{dt}$ as a function of y, determine all equilibrium solutions, and classify each as unstable or asymptotically stable.

A2. Consider the differential equation $\dfrac{dy}{dt} = (4-y)^2$.

(a) Explain why the solution $y(t) = 4$ is is called a *semi-stable* equilibrium solution.

(b) Plot $\dfrac{dy}{dt}$ as a function of y.

(c) Solve the initial-value problem $\dfrac{dy}{dt} = (4-y)^2; \quad y(0) = y_0$.

(d) Use your solution to show that if $y_0 < 4$, then $\lim_{t\to\infty} y(t) = 4$. What happens if $y_0 > 4$?

B1. For each condition below, construct a differential equation of the form $y' = f(y)$ satisfying the condition.

(a) The differential equation has exactly one equilibrium solution, and it is asymptotically stable.

(b) The differential equation has exactly two equilibrium solutions, and both are semistable.

(c) The differential equation has exactly two equilibrium solutions, and both are unstable.

B2. (Based on B & D §2.5 problems 16 and 17) Consider the Gompertz equation for population growth: $\dfrac{dy}{dt} = ry\ln\left(\dfrac{K}{y}\right)$ where K and r are positive constants. We consider only solutions with $y \geq 0$.

(a) Show that $y = 0$ and $y = K$ are equilibrium solutions to the Gompertz equation. (Caution: showing that $y = 0$ is an equilibrium is a little tricky.) Classify each as asymptotically stable or unstable.

(b) Produce a direction field for the Gompertz equation. (Use Maple if you want. Just pick some reasonable values for r and K.)

(c) Solve the Gompertz equation with initial condition $y(0) = y_0$. (Set $u = \ln(y/K)$ to get started. And remember: the chain rule is our friend.)

C1. (B & D §2.5 problem 20) The Schaefer model for a fish population $y(t)$ is given by $dy/dt = r(1 - y/K)y - Ey$, where r is a reproductive rate, K is a carrying capacity, and E is a measure of the effort put into harvesting the fish.

(a) Show that this differential equation has two equilibrium solutions, one at $y = 0$ and the other at $y = K(1 - E/r)$.

(b) Show that $y = 0$ is unstable and $y = K(1 - E/r)$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population $K(1 - E/r)$. Clearly the yield Y depends on E. The maximum value of $Y(E)$ is called the *maximum sustainable yield*. Find the maximum sustainable yield (it will depend on r and K), and the value of E that produces a maximum sustainable yield.

C2. (B & D §2.5 problem 23) Some diseases are spread largely by *carriers*, individuals who can transmit the disease, but who exhibit no overt symptoms. Let x and y, respectively, denote the proportion of susceptibles (that is, people who have the disease) and carriers in the population. Suppose that carriers are identified and removed from the population at a rate β, so $dy/dt = -\beta y$.

Suppose also that the disease spreads at a rate proportional to the product of x and y: $dx/dt = \alpha xy$.

(a) Solve the IVP $dy/dt = -\beta y$; $y(0) = y_0$ to determine the number of carriers present at any time t.

(b) Now solve the IVP $dx/dt = \alpha xy$; $x(0) = x_0$, remembering that x here is a function of t.

(c) Now find $\lim\limits_{t\to\infty} x(t)$, and interpret this result.