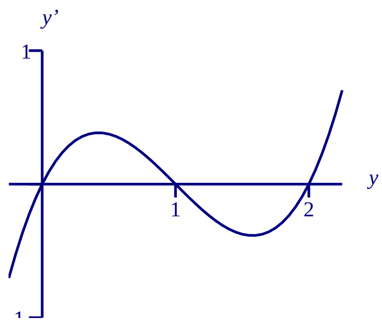
Math 333 Problem Set #3 – Solution 21 February 2003

A1. Consider the differential equation $\dfrac{dy}{dt} = y^3 - 3y^2 + 2y$. Plot $\dfrac{dy}{dt}$ as a function of y, determine all equilibrium solutions, and classify each as unstable or asymptotically stable.

Solution: Here is the picture:



The right-hand side factors as $y(y-2)(y-1)$, so there are equilibrium solutions at $y = 0$, $y = 1$, and $y = 2$. Solutions with $y < 0$ will decrease away from $y = 0$; solutions with $0 < y < 1$ will increase toward $y = 1$ (and away from $y = 0$), solutions with $1 < y < 2$ will decrease toward $y = 1$ (away from $y = 2$), and solutions with $y > 2$ will increase away from $y = 2$ Thus the solutions $y = 0$ and $y = 2$ are unstable and the solution $y = 1$ is asymptotically stable.

A2. Consider the differential equation $\dfrac{dy}{dt} = (4 - y)^2$.

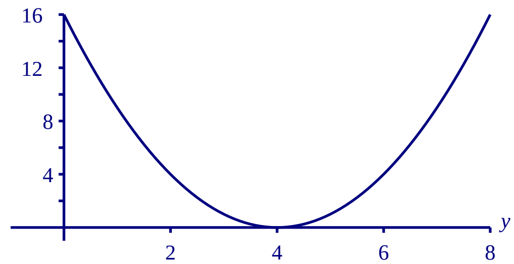
(a) Explain why the solution $y(t) = 4$ is is called a *semi-stable* equilibrium solution.

Solution: If $y(t) = 4$, then $dy/dt = (4-4)^2 = 0$. Thus the constant solution $y(t) = 4$ is an equilibrium solution.

If a solution $\varphi(t)$ to this differential equation takes on any value other than 4, then $(4 - \varphi(t))^2$ is positive, so that $d\varphi/dt$ is positive, and thus φ is increasing. Thus solutions that are less than $y = 4$ will increase to $y = 4$ and solutions that are greater than $y = 4$ will increase away from $y = 4$. The equilibrium $y = 4$ is semi-stable because some nearby solutions approach $y = 4$ as $t \to \infty$, and some diverge away.

(b) Plot $\dfrac{dy}{dt}$ as a function of y.

Solution: Here is the picture:



(c) Solve the initial-value problem $\dfrac{dy}{dt} = (4-y)^2; \quad y(0) = y_0$.

Solution: The equation is separable. We write

$$\frac{dy}{(4-y)^2} = dt$$

and integrate to get

$$\frac{1}{4-y} = t + C.$$

Using the initial condition $y(0) = y_0$, we have

$$\frac{1}{4-y_0} = C$$

so that

$$\frac{1}{4-y} = t + \frac{1}{4-y_0}.$$

We now solve for y, getting

$$y(t) = 4 - \frac{4-y_0}{t(4-y_0)+1}.$$

(d) Use your solution to show that if $y_0 < 4$, then $\lim_{t\to\infty} y(t) = 4$. What happens if $y_0 > 4$?

Solution: If $y_0 < 4$, then $4 - y_0$ is a positive number, so the denominator in $\dfrac{4-y_0}{t(4-y_0)+1}$ goes to ∞ as $t \to \infty$. Thus the fraction goes to zero, and the solution approaches 4.

If $y_0 > 4$, then $4 - y_0$ is a negative number, and the denominator $\dfrac{4-y_0}{t(4-y_0)+1}$ becomes zero when $t = 1/(y_0 - 4)$. So in this case, the solution $y(t)$ goes to infinity in finite time.

B1. For each condition below, construct a differential equation of the form $y' = f(y)$ satisfying the condition.

(a) The differential equation has exactly one equilibrium solution, and it is asymptotically stable.

Solution: The differential equation $y' = -y$ has the required property. The equilibrium solution $y = 0$ is asymptotically stable, because all solutions are of the form $y = Ce^{-t}$.

(b) The differential equation has exactly two equilibrium solutions, and both are semistable.

Solution: The differential equation $y' = (y^2 - 1)^2$ has this property. The equilibrium solutions at $y = -1$ and $y = 1$ are both semistable. This follows from the fact that $y'(t) \geq 0$ for all solutions, for all t, and that $y' = 0$ only at the equilibrium solutions $y = \pm 1$.

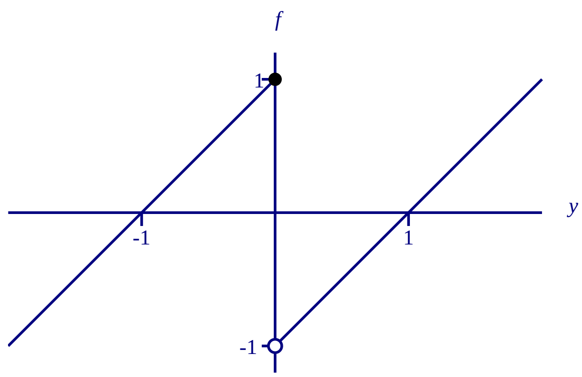
(c) The differential equation has exactly two equilibrium solutions, and both are unstable.

Solution: For this situation to occur, the function $f(y)$ must be discontinuous, because it must have two roots and must be increasing at each one. Here's such a function:

$$f(y) = y - \frac{|y|}{y}$$

with $f(0) = 1$.

Here's a graph of f.



The differential equation $y' = f(y)$ has equilibrium solutions at $y = -1$ and $y = 1$, and both are unstable.

B2. (Based on B & D §2.5 problems 16 and 17) Consider the Gompertz equation for population growth: $\dfrac{dy}{dt} = ry\ln\left(\dfrac{K}{y}\right)$ where K and r are positive constants. We consider only solutions with $y \geq 0$.

(a) Show that $y = 0$ and $y = K$ are equilibrium solutions to the Gompertz equation. (Caution: showing that $y = 0$ is an equilibrium is a little tricky.) Classify each as asymptotically stable or unstable.

Solution: To show that $y = 0$ is an equilibrium, we need to plug $y = 0$ in to $ry\ln\left(\dfrac{K}{y}\right)$. Unfortunately, this means dividing by zero, so we have to take a limit. Using l'Hospital's rule, we find that

$$\begin{aligned}
\lim_{y\to 0^+} ry\ln(K/y) &= r\lim_{y\to 0^+}\frac{\ln(K/y)}{1/y} \\
&= r\lim_{y\to 0^+}\frac{-\ln(y/K)}{1/y} \\
&= r\lim_{y\to 0^+}\frac{-\frac{1}{K}\cdot\frac{K}{y}}{-\frac{1}{y^2}} \\
&= r\lim_{y\to 0^+} y \\
&= 0.
\end{aligned}$$

So the right-hand side in the model approaches 0 as $y \to 0^+$, and we may consider $y = 0$ as an equilibrium solution.

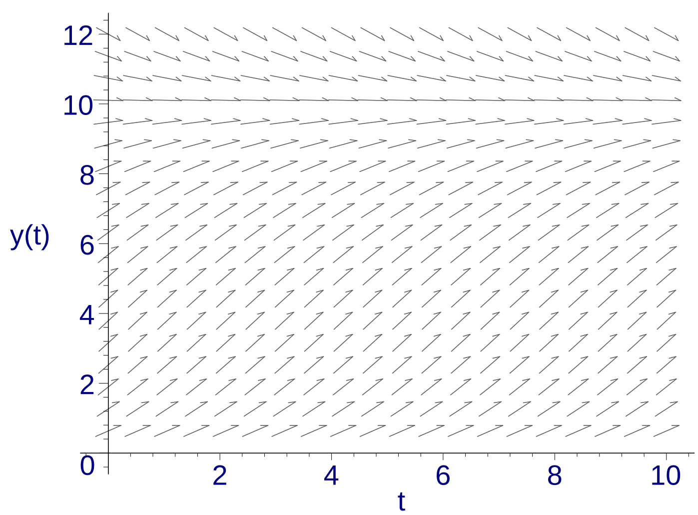
For $y = K$, the computation is much easier: since $\ln(K/K) = 0$, we get $dy/dt = 0$ for $y = K$.

There are no other equilibrium solutions, because either y or $\ln(K/y)$ must be zero at any equilibrium solution.

For $0 < y < K$, we know that y is positive and $K/y > 1$ so that $\ln(K/y)$ is positive. Thus dy/dt is positive. This makes $y = 0$ an unstable equilibrium. For $y > K$, we know that y is positive and $0 < K/y < 1$ so that $\ln(K/y)$ is negative. Thus dy/dt is positive for $y < K$ and negative for $y > K$. This makes $y = K$ an asymptotically stable equilibrium.

(b) Produce a direction field for the Gompertz equation. (Use Maple if you want. Just pick some reasonable values for r and K.)

Solution: Here is a picture. I used $K = 10$ and r is somewhere around 0.2.



(c) Solve the Gompertz equation with initial condition $y(0) = y_0$. (Set $u = \ln(y/K)$ to get started. And remember: the chain rule is our friend.)

Solution: Using the suggestion above, we set $u = \ln(y/K)$. First we notice that this makes $\ln(K/y)$ in the equation equal to $-u$. Then we find (using the chain rule) that

$$\frac{du}{dt} = \frac{1}{y}\frac{dy}{dt}.$$

Since the given equation has the form

$$\frac{1}{y}\frac{dy}{dt} = r\ln\left(\frac{K}{y}\right),$$

we can easily make the substitution to rewrite our equation as

$$\frac{du}{dt} = -ru.$$

We solve this by inspection, getting $u = Ce^{-rt}$ for some constant C. In fact, $C = u(0)$, which, by the definition of u above, is equal to $\ln(y_0/K)$. Undoing the original substitution, we have

$$\ln\left(\frac{y}{K}\right) = \ln\left(\frac{y_0}{K}\right)e^{-rt}.$$

Exponentiating both sides, we get

$$\frac{y}{K} = \left(\frac{y_0}{K}\right)^{e^{-rt}}$$

and solving for y, we get

$$y(t) = K\left(\frac{y_0}{K}\right)^{e^{-rt}}.$$

C1. (B & D §2.5 problem 20) The Schaefer model for a fish population $y(t)$ is given by $dy/dt = r(1-y/K)y - Ey$, where r is a reproductive rate, K is a carrying capacity, and E is a measure of the effort put into harvesting the fish.

(a) Show that this differential equation has two equilibrium solutions, one at $y = 0$ and the other at $y = K(1-E/r)$.
Solution: As a function of y, the expression $r(1-y/K)y - Ey$ is a quadratic, so it has at most two zeros. One of them is clearly $y = 0$; to find the other, we can factor out a y to get $y[r(1-y/K)-E]$ We set

$$r\left(1-\frac{y}{K}\right) - E = 0$$

and solve for y to get $y = K\left(1-\dfrac{E}{r}\right)$.

(b) Show that $y = 0$ is unstable and $y = K(1 - E/r)$ is asymptotically stable.

Solution: Since the coefficient of y^2 in $r(1-y/K)y - Ey$ is negative, we know that dy/dt as a function of y is a downward-opening parabola. From this, we know that the smaller of the equilibrium solutions (that is, $y = 0$) is unstable, and the larger of the two (that is, $y = K(1 - E/r)$) is stable. This is because solutions less than 0 will decrease away from zero; solutions slightly greater than zero will increase away from zero; solutions slightly less than the other equilibrium will increase; and solutions greater than the positive equilibrium will decrease.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population $K(1 - E/r)$. Clearly the yield Y depends on E. The maximum value of $Y(E)$ is called the *maximum sustainable yield*. Find the maximum sustainable yield (it will depend on r and K), and the value of E that produces a maximum sustainable yield.

Solution: As a function of E, we have

$$\begin{aligned} Y &= KE\left(1 - \frac{E}{r}\right) \\ &= K\left(E - \frac{E^2}{r}\right), \end{aligned}$$

a quadratic function with roots at $E = 0$ and $E = r$. There is a global maximum, which must be exactly half way between the roots. That is, $E = \frac{r}{2}$ will give the maximum sustainable yield of

$$Y\left(\frac{r}{2}\right) = \frac{Kr}{4}.$$

C2. (B & D §2.5 problem 23) Some diseases are spread largely by *carriers*, individuals who can transmit the disease, but who exhibit no overt symptoms. Let x and y, respectively, denote the proportion of susceptibles (that is, people who have the disease) and carriers in the population. Suppose that carriers are identified and removed from the population at a rate β, so $dy/dt = -\beta y$.

Suppose also that the disease spreads at a rate proportional to the product of x and y: $dx/dt = \alpha xy$.

(a) Solve the IVP $dy/dt = -\beta y; \quad y(0) = y_0$ to determine the number of carriers present at any time t.

Solution: We can solve the differential equation

$$\frac{dy}{dt} = -\beta y$$

by inspection. The solution is $y = Ce^{-\beta t}$ for an arbitrary constant C. Given the initial condition $y(0) = y_0$, we find that $C = y_0$, so the population of carriers is given by

$$y(t) = y_0 e^{-\beta t}.$$

(b) Now solve the IVP $dx/dt = \alpha xy; \quad x(0) = x_0$, remembering that x here is a function of t.

Solution: The differential equation for x now reads

$$\frac{dx}{dt} = \alpha x y_0 e^{-\beta t}.$$

This is linear, but it's also separable. We separate the expression on the right to get

$$\frac{dx}{x} = \alpha y_0 e^{-\beta t}\, dt.$$

Integrating both sides yields

$$\ln|x| = -\frac{\alpha y_0}{\beta} e^{-\beta t} + C.$$

Since x is a proportion of the population, x cannot be negative, so we have

$$\ln x = -\frac{\alpha y_0}{\beta} e^{-\beta t} + C,$$

or

$$\begin{aligned} x(t) &= \exp\left(-\frac{\alpha y_0}{\beta} e^{-\beta t} + C\right) \\ &= K \exp\left(-\frac{\alpha y_0}{\beta} e^{-\beta t}\right) \end{aligned}$$

for some positive constant K. Using the initial condition $x(0) = x_0$, we find

$$x_0 \;=\; K \exp\left(-\frac{\alpha y_0}{\beta}\right),$$

so that $K = x_0 \exp(\alpha y_0/\beta)$, and our final solution is

$$x(t) \;=\; x_0 \exp\left(\frac{\alpha y_0}{\beta}(1 - e^{-\beta t})\right).$$

(c) Now find $\lim_{t\to\infty} x(t)$, and interpret this result.

Solution: Since β is positive, as t goes to infinity, $e^{-\beta t}$ goes to zero, and the solution $x(t)$ approaches the constant

$$x_0 \exp\left(\frac{\alpha y_0}{\beta}\right).$$

Now, after examining the model for a while, we determine that it doesn't make sense as presented. The most reasonable way to repair the model is to say that $x(t)$ represents the porportion of the population that is still *healthy*, and that $dx/dt = -\alpha xy$, where α is a positive constant. That is, an encounter between a healthy individual and a carrier may result in a *decrease* in the healthy population.

The only algebraic change is the sign of α; our solution becomes

$$x(t) \;=\; x_0 \exp\left(-\frac{\alpha y_0}{\beta}(1 - e^{-\beta t})\right),$$

and we get

$$\lim_{t\to\infty} x(t) \;=\; x_0 \exp\left(-\frac{\alpha y_0}{\beta}\right).$$

This is the proportion of the population that remains healthy after a long time has elapsed.

We note that the exponent in this expression is negative, so that the proportion of healthy individuals in the long run is smaller than the initial proportion of healthy individuals. How much smaller depends directly on α (the infectuousness rate) and y_0 (the initial population of carriers), and inversely on β (the rate at which carriers are eliminated).