Reading: B & D, §2.7, 2.9

Exercises: Write your solutions clearly, and in complete sentences, remembering that they will be graded for presentation as well as correctness.

A1. Consider the initial value problem $y' = t - \frac{y}{3}$; $y(0) = 1$.

(a) Use Euler's method with $h = 0.2$ and $h = 0.1$ to find approximate solutions on the interval $0 \leq t \leq 2$. (You'll probably want to do this in Excel. Hand in printouts of the spreadsheets.)

(b) Solve the initial value problem, and find the exact values of $y(1)$ and $y(2)$.

(c) Compare the exact values to your estimates for $y(1)$ and $y(2)$. Compute the percentage errors for your estimates. Do the approximations appear to get better or worse as h decreases? As t increases?

A2. Consider the initial value problem $y' = ry$; $y(0) = 1$, where r is a constant. The (exact) solution to this problem is clearly $y = e^{rt}$.

(a) Let n be a positive integer. Apply Euler's method with $h = 1/n$ to find an approximate solution. Compute y_1, y_2, and y_3.

(b) Find a simple formula for y_k. (Hint: it's $(1 + r/n)^k$.) Optional: Prove that your formula is correct.

(c) We'd like to believe that as $h \to 0$, the approximation produced by Euler's method will approach the exact solution to our initial value problem. To see if this is the case, compute $\lim_{n\to\infty} y_n$ and compare it with the value of the exact solution to the IVP at $t = 1$.

B1. (B & D, §2.7, problem 15) Consider the initial value problem $y' = \frac{3t^2}{3y^2 - 4}$; $y(1) = 0$.

(a) Use Euler's method with $h = 0.1$ to obtain an approximate solution on $1 \leq t \leq 1.8$.

(b) Use Euler's method with $h = 0.05$ to obtain an approximate solution on $1 \leq t \leq 1.8$.

(c) The results in parts B1a and B1b agree quite closely at $t = 1.2$, 1.4, and 1.6, but you'll find that they are quite different at $t = 1.8$. Note (from the differential equation) that the line tangent to a solution curve is vertical when $y = \pm 2/\sqrt{3}$. Explain how this might cause such a difference in calculated values.

B2. Consider a mortgage in which an initial amount y_0 is borrowed. The borrower pays back k dollars per month, and the lender charges interest once at month at an annual rate r (assumed to be greater than 0). For each integer $n \geq 0$, let y_n denote the amount still owed n months into the mortgage.

(a) Write a difference equation for y_n. It should have the same form as equation (12) on p. 117 of B & D, so you can easily write down the solution to your difference equation; it's equation (14) on the same page.

(b) Suppose $y_0 = \$100,000$, $r = 6\%$, and the period of the mortgage is 20 years (that is, 240 months). Find the amount of the monthly payment.

(c) Suppose again that $y_0 = \$100,000$, $r = 6\%$, and that the borrower pays off the mortgage at $\$1,000$ per month. In how many months will the borrower have paid off the mortgage?

C1. (B & D §2.9, problem 16) Take $\rho > 1$ in the difference equation $u_{n+1} = \rho u_n(1 - u_n)$

(a) Draw a qualitatively correct stairstep diagram to determine $\lim_{n\to\infty} u_n$ if $u_0 < 0$.

(b) Draw a qualitatively correct stairstep diagram to determine what happens when $u_0 > 1$.

C2. Consider the difference equation $u_{n+1} = \rho u_n(1 - u_n)$

(a) When $\rho = 3.2$, the difference equation has a stable 2-cycle. That is, there are two numbers x_1 and x_2 such that the sequence $x_1, x_2, x_1, x_2, \ldots$ is a solution to the difference equation. Use a calculator or computer to estimate the values of x_1 and x_2. The easy way to do this is to pick an arbitrary value for u_0 (between 0 and 1) and start the difference equation running. Unless you're very unlucky, the solution you generate will approach the stable 2-cycle.

(b) The numbers x_1 and x_2 in C2a satisfy the equations

$$\begin{aligned} x_2 &= \rho x_1(1 - x_1) \\ x_1 &= \rho x_2(1 - x_2). \end{aligned}$$

Use the first equation to make a substitution for x_2 in the second, and thereby find a quartic equation for x_1. (By symmetry, x_2 will also be a root of your equation.)

(c) Set $\rho = 3.2$ and try to solve the equation in C2b. One of the roots will be zero; find the other three either by estimating them (with a calculator or computer) or, if you can, by solving the resulting cubic exactly. Verify that your estimates for x_1 and x_2 above agree with two of the roots.

(d) When $\rho = 3.48$, the difference equation has a stable 4-cycle. Use a calculator or computer to estimate the numbers in this 4-cycle (as in C2a).