A1. Consider the initial value problem $y' = t - \frac{y}{3}$; $y(0) = 1$.

(a) Use Euler's method with $h = 0.2$ and $h = 0.1$ to find approximate solutions on the interval $0 \le t \le 2$. (You'll probably want to do this in Excel. Hand in printouts of the spreadsheets.)

Solution: Here are the two Excel tables:

t_n	y_n	$f(t_n, y_n)$
0	1	−0.333333
0.2	0.933333	−0.111111
0.4	0.911111	0.096296
0.6	0.930370	0.289876
0.8	0.988345	0.470551
1	1.082455	0.639181
1.2	1.210292	0.796569
1.4	1.369606	0.943464
1.6	1.558299	1.080566
1.8	1.774412	1.208529
2	2.016118	1.327960

t_n	y_n	$f(t_n, y_n)$
0	1	−0.333333
0.1	0.966666	−0.222222
0.2	0.944444	−0.114814
0.3	0.932962	−0.010987
0.4	0.931864	0.089378
0.5	0.940802	0.186399
0.6	0.959441	0.280186
0.7	0.987460	0.370846
0.8	1.024545	0.458484
0.9	1.070393	0.543202
1	1.124713	0.625095
1.1	1.187223	0.704258
1.2	1.257649	0.780783
1.3	1.335727	0.854757
1.4	1.421203	0.926265
1.5	1.513830	0.995389
1.6	1.613369	1.062210
1.7	1.719590	1.126803
1.8	1.832270	1.189243
1.9	1.951194	1.249601
2	2.076154	1.307948

(b) Solve the initial value problem, and find the exact values of $y(1)$ and $y(2)$.

Solution: We write $y' + \frac{y}{3} = t$ and use the integrating factor $e^{\frac{t}{3}}$ to get

$$\frac{d}{dt}\left(e^{\frac{t}{3}}y\right) = te^{\frac{t}{3}}.$$

We integrate to get

$$ye^{\frac{t}{3}} \;=\; (3t-9)e^{\frac{t}{3}} + C,$$

so that

$$y \;=\; 3t - 9 + Ce^{-\frac{t}{3}}.$$

Using the initial condition, we find that

$$1 \;=\; -9 + C,$$

so that $C = 10$, and the solution to the initial value problem is

$$y \;=\; 3t - 9 + 10e^{-\frac{t}{3}}.$$

The exact values of $y(1)$ and $y(2)$ are, respectively, $10e^{-\frac{1}{3}} - 6$ and $10e^{-\frac{2}{3}} - 3$.

(c) Compare the exact values to your estimates for $y(1)$ and $y(2)$. Compute the percentage errors for your estimates. Do the approximations appear to get better or worse as h decreases? As t increases?

Solution: To six digits, we have $y(1) = 1.165313$ and $y(2) = 2.134171$. We subtract these from the values for $y(1)$ and $y(2)$ in the tables above and divide the difference by the actual values to get the following percentage errors. (In each case, the Euler approximation is less than the actual value.)

	h	
t	0.2	0.1
1	7.11%	3.48%
2	5.53%	2.72%

It appears that the errors decrease as h decreases and as t increases. This suggests that the solutions in this problem are converging. (In fact, we know they are. From the form of the exact solution, it's clear that every solution to this differential equation is asymptotic to the line $y = 3t - 9$.)

A2. Consider the initial value problem $y' = ry$; $y(0) = 1$, where r is a constant. The (exact) solution to this problem is clearly $y = e^{rt}$.

(a) Let n be a positive integer. Apply Euler's method with $h = 1/n$ to find an approximate solution. Compute y_1, y_2, and y_3.

Solution: In this differential equation, we have $f(t, y) = ry$, so the iterative step in Euler's method is simply

$$\begin{aligned} y_{n+1} &= y_n + hry_n \\ &= y_n + \frac{r}{n} y_n \\ &= \left(1 + \frac{r}{n}\right) y_n. \end{aligned}$$

Thus we get

$$\begin{aligned} y_1 &= \left(1 + \frac{r}{n}\right) y_0 \\ &= \left(1 + \frac{r}{n}\right) \\ y_2 &= \left(1 + \frac{r}{n}\right) y_1 \\ &= \left(1 + \frac{r}{n}\right)^2 \\ y_3 &= \left(1 + \frac{r}{n}\right) y_2 \\ &= \left(1 + \frac{r}{n}\right)^3. \end{aligned}$$

(b) Find a simple formula for y_k. (Hint: it's $(1 + r/n)^k$.) Optional: Prove that your formula is correct.

Solution: The calculations above suggest that $y_k = \left(1 + \frac{r}{n}\right)^k$ for each integer $k \geq 0$. To prove that this is correct we proceed by induction on k.

For the base case, we observe that $y_0 = 1$ and that $\left(1 + \frac{r}{n}\right)^0 = 1$ as well (assuming that $r/n \neq -1$).

For the inductive step, assume that $k \geq 0$ and that $y_k = \left(1 + \frac{r}{n}\right)^k$. Then using the Euler's method iteration above, we get

$$y_{k+1} = \left(1 + \frac{r}{n}\right) y_k$$

$$
\begin{aligned}
&= \left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)^k \\
&= \left(1+\frac{r}{n}\right)^{k+1}
\end{aligned}
$$

as required. By the principle of mathematical induction, our formula holds for all $k \geq 0$.

(c) We'd like to believe that as $h \to 0$, the approximation produced by Euler's method will approach the exact solution to our initial value problem. To see if this is the case, compute $\lim_{n\to\infty} y_n$ and compare it with the value of the exact solution to the IVP at $t = 1$.

Solution: Let $A = \lim_{n\to\infty} y_n$. Then we have

$$
A = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n
$$

so that

$$
\begin{aligned}
\ln A &= \lim_{n\to\infty} n \ln\left(1+\frac{r}{n}\right) \\
&= \lim_{n\to\infty} \frac{\ln(1+r/n)}{(1/n)}.
\end{aligned}
$$

This last limit has the indeterminate form $0/0$, so we may apply l'Hospital's rule to get

$$
\begin{aligned}
\ln A &= \lim_{n\to\infty} \frac{(1+r/n)^{-1}(-r/n^2)}{(-1/n^2)} \\
&= \lim_{n\to\infty} r(1+r/n)^{-1} \\
&= r.
\end{aligned}
$$

Thus $A = e^r$. That is, as $n \to \infty$ (or equivalently, as $h \to 0$), the last step in the Euler's method solution to this problem approaches e^r. The exact solution also satisfies $y(1) = e^r$, so in this case at least, Euler's method converges to the exact answer as $h \to 0$.

B1. (B & D, §2.7, problem 15) Consider the initial value problem $y' = \dfrac{3t^2}{3y^2-4}$; $y(1) = 0$.

(a) Use Euler's method with $h = 0.1$ to obtain an approximate solution on $1 \leq t \leq 1.8$.

Solution: Here is the Excel spreadsheet:

t_n	y_n	$f(t_n, y_n)$
1	0	-0.75
1.1	-0.075	-0.911344
1.2	-0.166134	-1.102829
1.3	-0.276417	-1.344549
1.4	-0.410872	-1.683100
1.5	-0.579182	-2.254777
1.6	-0.804660	-3.732565
1.7	-1.177916	53.365876
1.8	4.158670	0.202992

(b) Use Euler's method with $h = 0.05$ to obtain an approximate solution on $1 \leq t \leq 1.8$.

Solution: Here is the Excel spreadsheet:

t_n	y_n	$f(t_n, y_n)$
1	0	-0.75
1.05	-0.0375	-0.827748
1.1	-0.078887	-0.911755
1.15	-0.124475	-1.003536
1.2	-0.174652	-1.105286
1.25	-0.229916	-1.220253
1.3	-0.290928	-1.353414
1.35	-0.358599	-1.512775
1.4	-0.434238	-1.712133
1.45	-0.519845	-1.977717
1.5	-0.618731	-2.367162
1.55	-0.737089	-3.041012
1.6	-0.889139	-4.716599
1.65	-1.124969	-40.168825
1.7	-3.133411	0.340603
1.75	-3.116380	0.365519
1.8	-3.098104	0.392018

(c) The results in parts B1a and B1b agree quite closely at $t = 1.2$, 1.4, and 1.6, but you'll find that they are quite different at $t = 1.8$. Note (from the differential equation) that the line tangent to a solution curve is vertical when $y = \pm 2/\sqrt{3}$. Explain how this might cause such a difference in calculated values.

Solution: The two approximations fly apart at about $t = 1.65$, $y = -1.15$ (give or take). The striking thing is that the values for y' (that is, $f(t, y)$) are wildly different at this point. In particular, one is large and positive, and the other is large and negative.

Clearly this is what causes the two approximations to fly apart.

Why does it happen? For values of y close to $-2/\sqrt{3} \approx -1.15$, the denominator in $f(t, y)$ is close to zero, so the Euler's method slopes will be very large. If y is a little less than -1.15, then the denominator in $f(t, y)$ will be small and positive, so the Euler's method slope will be large and positive; if y is a little greater than -1.15, then the denominator in $f(t, y)$ will be small and negative, so the Euler's method slope will be large and negative.

Even though the two values of y are close together, the calculated slopes are very different, because they lie on opposite sides of a place where $f(t, y)$ has a zero denominator.

B2. Consider a mortgage in which an initial amount y_0 is borrowed. The borrower pays back k dollars per month, and the lender charges interest once at month at an annual rate r (assumed to be greater than 0). For each integer $n \geq 0$, let y_n denote the amount still owed n months into the mortgage.

(a) Write a difference equation for y_n. It should have the same form as equation (12) on p. 117 of B & D, so you can easily write down the solution to your difference equation; it's equation (14) on the same page.

Solution: The difference equation is

$$y_{n+1} \quad = \quad y_n \left(1 + \frac{r}{12}\right) - k.$$

This is just equation (12) from the book with $\rho = 1 + \dfrac{r}{12}$ and $b = -k$. Since $\rho \neq 1$, we can use the solution given in line (14). It's

$$y_n \quad = \quad y_0 \left(1 + \frac{r}{12}\right)^n - \left(\frac{1 - (1 + r/12)^n}{1 - (1 + r/12)}\right) k.$$

We can simplify this a little. We get

$$y_n = \left(1+\frac{r}{12}\right)^n y_0 - \left[\left(1+\frac{r}{12}\right)^n - 1\right]\frac{12k}{r}.$$

(b) Suppose $y_0 = \$100,000$, $r = 6\%$, and the period of the mortgage is 20 years (that is, 240 months). Find the amount of the monthly payment.

We set $y_{240} = 0$ and solve the resulting equation for k. We get

$$\begin{aligned}\left(1+\frac{r}{12}\right)^{240} y_0 &= \left[\left(1+\frac{r}{12}\right)^{240} - 1\right]\frac{12k}{r} \\ k &= \frac{ry_0}{12}\frac{\left(1+\frac{r}{12}\right)^{240}}{\left(1+\frac{r}{12}\right)^{240}-1}\end{aligned}$$

Putting in the given values for r and y_0, we get a monthly payment of about \$716.43.

(c) Suppose again that $y_0 = \$100,000$, $r = 6\%$, and that the borrower pays off the mortgage at \$1,000 per month. In how many months will the borrower have paid off the mortgage?

We need to solve

$$0 = \left(1+\frac{r}{12}\right)^n y_0 - \left[\left(1+\frac{r}{12}\right)^n - 1\right]\frac{12k}{r}$$

for n. After some algebraic manipulation, we find that the equation above is equivalent to

$$\left(1+\frac{r}{12}\right)^n = \frac{12k}{12k - ry_0}$$

so that the solution is

$$n = \frac{\ln\left(\frac{12k}{12k-ry_0}\right)}{\ln\left(1+\frac{r}{12}\right)}$$

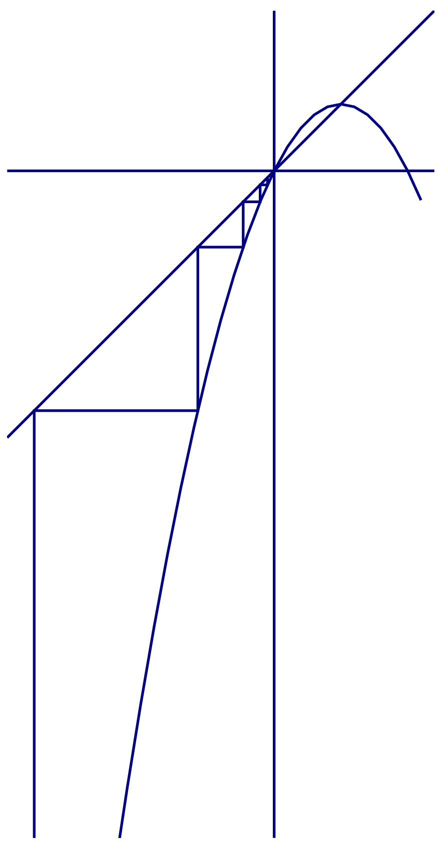
Putting in the given values for r, k, and y_0, we find that $n \approx 138.976$, so that the mortgage will be paid off in 139 months.

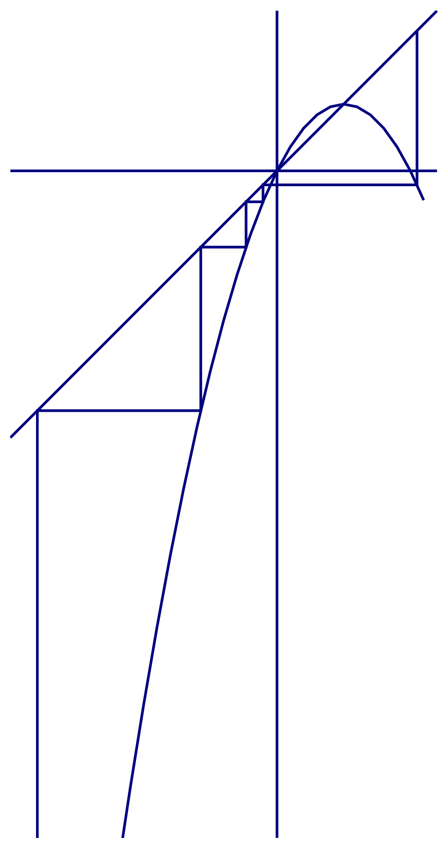
C1. (B & D §2.9, problem 16) Take $\rho > 1$ in the difference equation $u_{n+1} = \rho u_n(1 - u_n)$

(a) Draw a qualitatively correct stairstep diagram to determine $\lim_{n\to\infty} u_n$ if $u_0 < 0$.

(b) Draw a qualitatively correct stairstep diagram to determine what happens when $u_0 > 1$.

Here are two diagrams, each with $\rho = 2$. In the first, I took $u_0 = -0.05$ and in the second I took $u_0 = 1.05$. In each case, it's clear that u_n is going to $-\infty$.





C2. Consider the difference equation $u_{n+1} = \rho u_n(1 - u_n)$

(a) When $\rho = 3.2$, the difference equation has a stable 2-cycle. That is, there are two numbers x_1 and x_2 such that the sequence $x_1, x_2, x_1, x_2, \ldots$ is a solution to the difference equation. Use a calculator or computer to estimate the values of x_1

and x_2. The easy way to do this is to pick an arbitrary value for u_0 (between 0 and 1) and start the difference equation running. Unless you're very unlucky, the solution you generate will approach the stable 2-cycle.

Solution: We choose an arbitrary value for x on the calculator, and then repeatedly store $3.2x(1-x)$ back into x. The sequence very quickly settles down to the two values

$$x_1 \approx 0.513045 \quad \text{and} \quad x_2 \approx 0.799455.$$

(b) The numbers x_1 and x_2 in C2a satisfy the equations

$$\begin{aligned} x_2 &= \rho x_1(1-x_1) \\ x_1 &= \rho x_2(1-x_2). \end{aligned}$$

Use the first equation to make a substitution for x_2 in the second, and thereby find a quartic equation for x_1. (By symmetry, x_2 will also be a root of your equation.)

Solution: Making the substitution, we get

$$\begin{aligned} x_1 &= \rho(\rho x_1(1-x_1)(1-\rho x_1(1-x_1)) \\ &= \rho^2 x_1(1-x_1)(1-\rho x_1-\rho x_1^2) \\ &= \rho^2 x_1(1-(\rho+1)x_1+2\rho x_1^2-\rho x_1^3) \\ &= \rho^2 x_1-\rho^2(\rho+1)x_1^2+2\rho^3 x_1^3-\rho^3 x_1^4. \end{aligned}$$

Moving everything to one side, we get

$$\rho^3 x_1^4 - 2\rho^3 x_1^3 + \rho^2(\rho+1)x_1^2 + (1-\rho^2)x_1 \;=\; 0.$$

(c) Set $\rho = 3.2$ and try to solve the equation in C2b. One of the roots will be zero; find the other three either by estimating them (with a calculator or computer) or, if you can, by solving the resulting cubic exactly. Verify that your estimates for x_1 and x_2 above agree with two of the roots.

Solution: We rewrite the polynomial without the subscript on x. Our task is to find the solutions of the quartic equation

$$\rho^3 x^4 - 2\rho^3 x^3 + \rho^2(\rho+1)x^2 + (1-\rho^2)x \;=\; 0.$$

Clearly we can factor out an x to get

$$x(\rho^3x^3 - 2\rho^3x^2 + \rho^2(\rho+1)x + (1-\rho^2)) \;=\; 0.$$

We asked *Mathematica* to factor the resulting cubic, and it found that there is a linear factor. We get

$$x(\rho x - (\rho-1))(\rho^2x^2 - \rho(\rho+1)x + (\rho+1)) \;=\; 0,$$

so that two of the roots of our quartic are $x = 0$ and $x = \dfrac{\rho-1}{\rho}$. It is not a coincidence that these are the two equilibria (both unstable) in the logistic difference equation. An equilibrium solution would have to satisfy both

$$x_1 = \rho x_1(1-x_1) \quad \text{and} \quad x_2 = \rho x_2(1-x_2).$$

We can find the remaining two roots by using the quadratic formula. They are

$$\begin{aligned} \frac{\rho(\rho+1) \pm \sqrt{\rho^2(\rho+1)^2 - 4\rho^2(\rho+1)}}{2\rho^2} &= \frac{\rho(\rho+1) \pm \sqrt{\rho^2(\rho+1)(\rho-3)}}{2\rho^2} \\ &= \frac{\rho+1 \pm \sqrt{(\rho+1)(\rho-3)}}{2\rho}. \end{aligned}$$

When we substitute $\rho = 3.2$ into this expression we get $\dfrac{4.2 \pm \sqrt{0.84}}{8.4}$, which simplifies to $\dfrac{21 \pm \sqrt{21}}{32}$. These are the exact values of the numbers we estimated with the calculator in the first part of this problem.

(d) When $\rho = 3.48$, the difference equation has a stable 4-cycle. Use a calculator or computer to estimate the numbers in this 4-cycle (as in C2a).

Solution: To seven digits, the calculator settles down on the sequence

$$0.3950645, \quad 0.8316801, \quad 0.4871593, \quad 0.8694262$$