A1. The solutions to the following four second-order initial value problems are all so common that they have their own names. For each problem below, find the solution in terms of exponential functions (possibly with imaginary exponents), and then give the solution's more common name. (You'll recognize them – for example, the common name of the solution to the first problem is cosine.)

(a) $y'' + y = 0; \quad y(0) = 1, \quad y'(0) = 0.$

(b) $y'' + y = 0; \quad y(0) = 0, \quad y'(0) = 1.$

(c) $y'' - y = 0; \quad y(0) = 1, \quad y'(0) = 0.$

(d) $y'' - y = 0; \quad y(0) = 0, \quad y'(0) = 1.$

Solution:

(a) The characteristic equation is $r^2+1 = 0$, which has roots $\pm i$. The general solution is

$$y = c_1 e^{it} + c_2 e^{-it}.$$

From this we get $y' = c_1 ie^{it} - c_2 ie^{it}$. The initial conditions imply that

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ ic_1 & - & ic_2 & = & 0. \end{array}$$

The second equation implies that $c_1 = c_2$, and so from the first we get $c_1 = c_2 = \frac{1}{2}$. In terms of exponents, the solution is

$$y = \frac{e^{it} + e^{-it}}{2}.$$

This function is also known as $\cos(t)$.

(b) The situation is just as in the previous part, except that the equations for c_1 and c_2 become

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ ic_1 & - & ic_2 & = & 1. \end{array}$$

This time, the first equation implies that $c_2 = -c_1$, so that the second equation becomes $ic_1 + ic_1 = 1$, so that $c_1 = \frac{1}{2i}$ and $c_2 = -\frac{1}{2i}$. In terms of exponents, the solution is

$$y = \frac{e^{it} - e^{-it}}{2i}.$$

This function is also known as $\sin(t)$.

(c) The characteristic equation is $r^2 - 1 = 0$, which has roots ± 1. The general solution to the differential equation is

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

From this we get $y'(t) = c_1 e^t - c_2 e^{-t}$. The initial conditions imply that

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 0. \end{aligned}$$

The second equation says $c_1 = c_2$, and thus from the first equation we get $c_1 = c_2 = \frac{1}{2}$. The solution is

$$y(t) = \frac{e^t + e^{-t}}{2}.$$

This function is also known as $\cosh(t)$.

(d) As in the previous part, the general solution is $y = c_1 e^t + c_2 e^{-t}$. The system of equations for c_1 and c_2 is

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 1. \end{aligned}$$

The first equation says that $c_1 = -c_2$. Substituting into the second equation, we get $2c_1 = 1$, so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The solution is then

$$y(t) = \frac{e^t - e^{-t}}{2}.$$

This function is known familiarly as $\sinh(t)$.

B1. Consider the initial value problem

$$y'' + 3y' - 4y = 0; \quad y(0) = 1, \quad y'(0) = \alpha.$$

(a) Solve the initial value problem. Your solution will depend on the parameter α.

(b) The behavior of the solution as $t \to \infty$ also depends on α.

  i. For what values of α does the solution to the initial value problem go to $+\infty$ as $t \to \infty$? (Your answer should be an interval in the real line.)

  ii. For what values of α does the solution go to $-\infty$?

  iii. For what values of α does the solution remain bounded? What is the limiting value?

Solution:

(a) The characteristic equation is $r^2 + 3r - 4 = 0$. We can factor this to get

$$(r+4)(r-1) = 0,$$

so the roots are $r = -4$ and $r = 1$. The general solution to the differential equation is

$$y(t) = c_1 e^{-4t} + c_2 e^{t}.$$

From this we get

$$y'(t) = -4c_1 e^{-4t} + c_2 e^{t}.$$

The initial conditions then give us the system

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 1 \\ -4c_1 & + & c_2 & = & \alpha. \end{array}$$

From the top equation we get $c_1 = 1 - c_2$. Substituting this into the second equation yields

$$\begin{aligned} \alpha &= c_2 - 4(1 - c_2) \\ &= 5c_2 - 4 \end{aligned}$$

So $c_2 = \dfrac{4+\alpha}{5}$. Thus $c_1 = 1 - c_2 = \dfrac{1-\alpha}{5}$. The solution to the initial value problem is

$$y(t) = \frac{1-\alpha}{5} e^{-4t} + \frac{4+\alpha}{5} e^{t}.$$

(b) As $t \to \infty$, the e^{-4t} term goes to zero, and the expression e^t goes to infinity. The coefficient of e^t will determine the asymptotic behavior of the entire solution.

i. For $\alpha > -4$, the coefficient of e^t is positive, so the solution will go to $+\infty$ as $t \to \infty$.

ii. For $\alpha < -4$, the coefficient of the e^t term is negative, and the solution goes to $-\infty$ as $t \to \infty$.

iii. If the solution is to remain bounded as $t \to \infty$, the coefficient of e^t must be zero. This occurs only when $\alpha = -4$. If $\alpha = -4$, then the solution is simply $y(t) = e^{-4t}$, which approaches 0 as $t \to \infty$.

B2. (a) Let α denote a positive constant. Solve the initial value problem

$$2y'' + 5y' - 3y = 0; \quad y(0) = 3,\ y'(0) = -\alpha.$$

(b) Now let $\alpha = 2$ in your solution to part B2a, and show that the solution has a global minimum. Find the value of t where the global minimum occurs. (Give the exact value.)

(c) What is the smallest value of α for which your solution in part (B2a) has no local minimum?

Solution:

(a) The characteristic equation for this differential equation is

$$\begin{aligned} 0 &= 2r^2 + 5r - 3 \\ &= (2r-1)(r+3), \end{aligned}$$

so a general solution is

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-3t}.$$

The derivative of this solution is

$$y' = \frac{1}{2} c_1 e^{\frac{1}{2}t} - 3c_2 e^{-3t}$$

and so the initial conditions become

$$\begin{aligned} 3 &= c_1 + c_2 \\ -\alpha &= \frac{1}{2}c_1 - 3c_2. \end{aligned}$$

The solution to this system is

$$c_1 = \frac{18 - 2\alpha}{7} \qquad c_2 = \frac{3 + 2\alpha}{7}.$$

The solution to the initial value problem is

$$y \;=\; \left(\frac{18 - 2\alpha}{7}\right) e^{\frac{1}{2}t} + \left(\frac{3 + 2\alpha}{7}\right) e^{-3t}. \tag{1}$$

(b) With $\alpha = 2$, our solution becomes

$$y \;=\; 2e^{\frac{1}{2}t} + e^{-3t}.$$

We note that this function is continuous and differentiable on the whole real line. We find that

$$y' \;=\; e^{\frac{1}{2}t} - 3e^{-3t}.$$

If there is a global minimum, it must occur at a critical point, that is at a value of t where

$$e^{\frac{1}{2}t} - 3e^{-3t} \;=\; 0.$$

We solve this equation for t, getting $t = \dfrac{2}{7} \ln 3$. Next we observe that

$$\lim_{t \to \infty} 2e^{\frac{1}{2}t} + e^{-3t} \;=\; \infty$$

and

$$\lim_{t \to -\infty} 2e^{\frac{1}{2}t} + e^{-3t} \;=\; \infty.$$

Since our solution goes to $+\infty$ as $t \to \pm\infty$ and it has only one critical point, that critical point must be a global minimum.

(c) We first note that if $0 < \alpha < 9$, then both coefficients in the solution (1) are positive, so that the solution tends to $+\infty$ as $t \to \pm\infty$. Next we calculate

$$y' \;=\; \frac{9 - \alpha}{7} e^{\frac{1}{2}t} - \frac{9 + 6\alpha}{7} e^{-3t}$$

and set $y'(t) = 0$ to locate any critical points. We find that $y'(t) = 0$ when

$$t = \frac{2}{7} \ln \left(\frac{9 + 6\alpha}{9 - \alpha} \right).$$

If $0 < \alpha < 9$, then the expression in parentheses is positive, so there is a single critical point, and that critical point must be a global minimum, by the argument above.

If $\alpha \geq 9$, on the other hand, then the expression in parentheses is negative, so there is no critical point, and therefore no global minimum. The smallest value of α for which there is no global minimum is $\alpha = 9$.

C1. (a) Assume that $|b| < 2$. Find the (real-valued) general solution to the differential equation

$$y'' + by' + y = 0.$$

You should get a family of solutions including two constants c_1 and c_2, and depending on the parameter b.

(b) Describe in qualitative terms (increasing, decreasing, oscillating, and so on) the general behavior of your family of solutions for

i. $b = 1$.

ii. $b = 0$.

iii. $b = -1$.

Plot a typical solution for each of the given values of b. (Choose any convenient values for c_1 and c_2 (except $c_1 = c_2 = 0$).)

(c) What happens to your solutions as b increases toward 2? What happens as b decreases toward -2?

Solution:

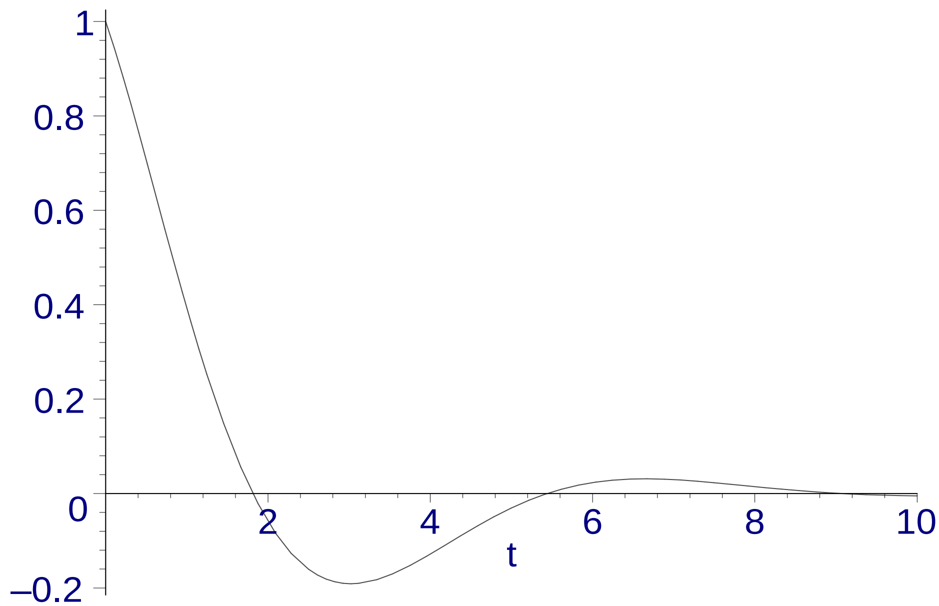
(a) The characteristic equation is $r^2 + br + 1 = 0$, so the roots are

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4}}{2} \\ &= -\frac{b}{2} \pm \frac{i}{2}\sqrt{4 - b^2}. \end{aligned}$$

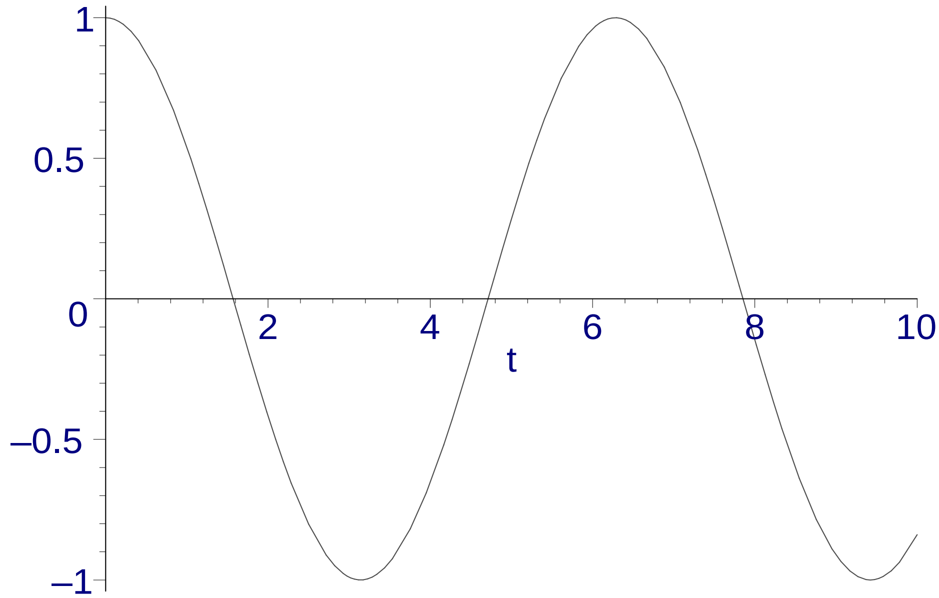
The general solution to the differential equation thus has the form

$$y = e^{-\frac{b}{2}t}\left(c_1 \cos\left(t\frac{\sqrt{4-b^2}}{2}\right) + c_2 \sin\left(t\frac{\sqrt{4-b^2}}{2}\right)\right).$$

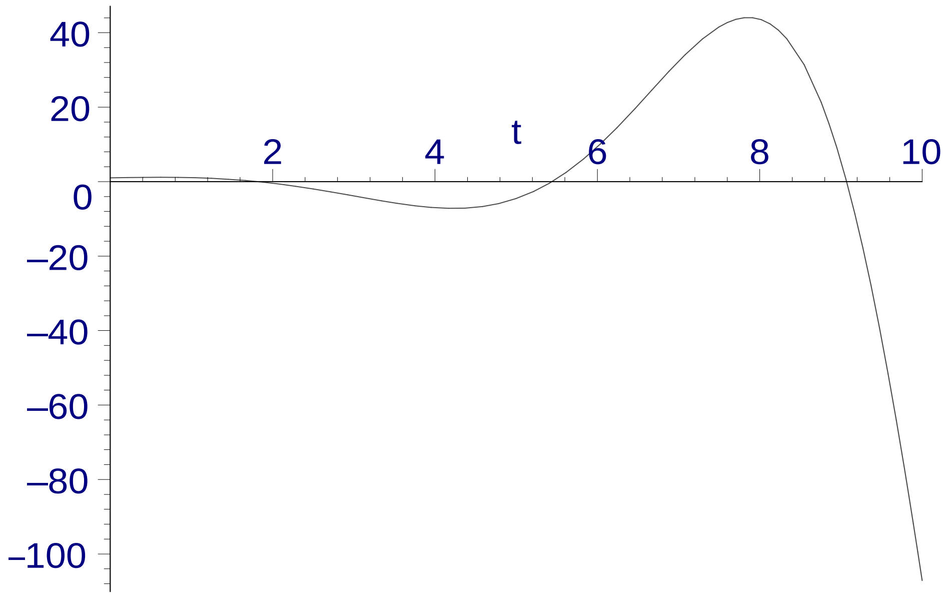
(b) i. At $b = 1$, every solution is an exponentially-decaying sinusoidal with angular velocity $\sqrt{3}/2$, and thus with period period $4\pi/\sqrt{3}$. Every solution approaches 0 as $t \to \infty$.
Here is a plot of the solution $e^{-t/2}\cos(\sqrt{3}t/2)$.



ii. At $b = 0$, every solution is a sinusoidal with period 2π and constant amplitude $\sqrt{c_1^2 + c_2^2}$.
Here is a plot of the solution $\cos(t)$.



iii. At $b = -1$, every solution is sinusoidal with angular velocity $\sqrt{3}/2$ (and thus period $4\pi/\sqrt{3}$) and exponentially-increasing amplitude.
Here is a plot of the solution $e^{t/2}\cos(\sqrt{3}t/2)$. The vertical scale is compressed.



(c) As b approaches ± 2, the period of the sinusoidal goes to infinity.

For b just less than 2, every solution is a sinusoid with very low frequency and with exponentially-decreasing amplitude. All solutions approach zero as $t \to \infty$, but also cross the $y = 0$ axis inifnitely many times.

For b just greater than -2, every solution is a sinusoid with very low frequency and exponentially-increasing amplitude. This is a difficult case to analyze, since the sinusoid (which increases and decreases very slowly) is being multiplied by an exponential that increases very rapidly.