A1. Solve the initial value problem $y'' + 4y' + 4y = 0; \quad y(0) = 1,\ y'(0) = 0.$

Solution: The characteristic equation is $r^2 + 4r + 4 = 0$, so the characteristic roots are -2 and -2. The general solution has the form

$$y = (c_1 + c_2 t)e^{-2t}.$$

From this we get

$$y' = -2(c_1 + c_2 t)e^{-2t} + c_2 e^{-2t}.$$

The initial conditions give us the system

$$\begin{aligned} c_1 \phantom{{}+c_2} &= 1 \\ -2c_1 + c_2 &= 0 \end{aligned}$$

from which we get $c_1 = 1$ and $c_2 = 2$. The solution is

$$y = (1 + 2t)e^{-2t}.$$

A2. Consider the third-order differential equation

$$y''' + 6y'' + 12y' + 8y = 0. \tag{1}$$

The characteristic equation of (1) is $(r + 2)^3 = 0$, and, as you can check, $y_1 = e^{-2t}$ is a solution.

(a) Find two other solutions, y_2 and y_3, that are not scalar multiples of y_1 or of each other.

(Probably the best way to do this is to make guesses and then plug them in to (1) to see if they work.)

Solution: We will guess that the other two solutions are $y_2 = te^{-2t}$ and $y_3 = t^2 e^{-2t}$. To check this, we compute

$$\begin{aligned} y_2' &= -2te^{-2t} + e^{-2t}, \\ y_2'' &= 4te^{-2t} - 4e^{-2t}, \\ y_2''' &= -8te^{-2t} + 12e^{-2t}; \\ y_3' &= -2t^2 e^{-2t} + 2te^{-2t}, \\ y_3'' &= 4t^2 e^{-2t} - 8te^{-2t} + 2e^{-2t}, \\ y_3''' &= -8t^2 e^{-2t} + 24te^{-2t} - 12e^{-2t}. \end{aligned}$$

Checking y_2 in the differential equation above, we get

$$\begin{aligned} y_2''' + 6y_2'' + 12y_2' + 8y_2 &= (-8t+12)e^{-2t} + 6(4t-4)e^{-2t} \\ &\quad +12(-2t+1)e^{-2t} + 8te^{-2t} \\ &= ((-8+24-24+8)t + (12-24+12))e^{-2t} \\ &= 0. \end{aligned}$$

Checking y_3, we get

$$\begin{aligned} y_3''' + 6y_3'' + 12y_3' + 8y_3 &= (-8t^2+24t-12)e^{-2t} + 6(4t^2-8t+2)e^{-2t} \\ &\quad +12(-2t^2+2t)e^{-2t} + 8t^2e^{-2t} \\ &= ((-8+24-24+8)t^2 + (24-48+24)t + (-12+12))e^{-2t} \\ &= 0. \end{aligned}$$

So our guesses were correct.

(b) Solve the third-order initial value problem

$$y''' + 6y'' + 12y' + 8y = 0; \qquad y(0) = 1, \quad y'(0) = -5, \quad y''(0) = 20.$$

Solution: The general solution is $y = (c_1t^2 + c_2t + c_3)e^{-2t}$. From this we get

$$\begin{aligned} y' &= (2c_1t + c_2)e^{-2t} - 2(c_1t^2 + c_2t + c_3)e^{-2t} \\ y'' &= 2c_1e^{-2t} - 2(2c_1t + c_2)e^{-2t} - 2(2c_1t + c_2)e^{-2t} + 4(c_1t^2 + c_2t + c_3)e^{-2t} \\ &= 2c_1e^{-2t} - 4(c_1t + c_2)e^{-2t} + 4(c_1t^2 + c_2t + c_3)e^{-2t}. \end{aligned}$$

Imposing the given initial conditions, we get the system

$$\begin{array}{rcrcrcr} & & & & c_3 & = & 1 \\ & & c_2 & - & 2c_3 & = & -5 \\ 2c_1 & - & 4c_2 & + & 4c_3 & = & 20 \end{array}$$

The solution to the system is $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, so the solution to the IVP is

$$y = (2t^2 - 3t + 1)e^{-2t}.$$

B1. (B & D, §3.5, problem 24) Given that $y_1 = t$ is one solution to the differential equation $t^2y'' + 2ty' - 2y = 0$, (with $t > 0$), use the method of reduction of order to find a second solution, y_2.

Then calculate $W(y_1, y_2)$ to be sure that y_2 is truly independent of y_1.

Solution: We guess that $y_2 = tv(t)$ is a second solution. We have

$$\begin{aligned} y_2' &= v(t) + tv'(t) \\ y_2'' &= 2v'(t) + tv''(t). \end{aligned}$$

Plugging these expressions into the given differential equation (and suppressing the argument of v), we get

$$\begin{aligned} 0 &= t^2(2v' + tv'') + 2t(v + tv') - 2tv \\ &= t^3v'' + 4t^2v' \end{aligned}$$

Since $t > 0$, we may divide through by t^2 to get

$$tv'' + 4v' = 0.$$

Next we let $u = v'$ so that $v'' = u'$. Making this substitution, we find that

$$tu' + 4u = 0.$$

We can solve this by the method of integrating factors. When we multiply through by t^3, we get

$$\begin{aligned} t^4u' + 4t^3u &= 0 \\ \frac{d}{dt}(t^4u) &= 0 \end{aligned}$$

so that $u = c_1t^{-4}$. From this, we get that

$$v = \int u\,dt = c_2t^{-3} + c_3.$$

We started by assuming that tv was a solution to the differential equation. We now have $tv = c_2t^{-2} + c_3t$. We already had the term c_3t; the new part of the solution is c_2t^{-2}. So we take $y_2 = t^{-2}$.

To verify that this in truly new, we compute the Wronskian. We get

$$\begin{aligned} W(t, t^{-2}) &= \begin{vmatrix} t & 1 \\ t^{-2} & -2t^{-3} \end{vmatrix} \\ &= -2t^{-2} - t^{-2} \\ &= -3t^{-2}. \end{aligned}$$

Since $t > 0$ in this problem, we know that the Wronskian is never 0. In particular, there is some value of t for which $W(t, t^{-2})(t) \neq 0$.

B2. Carry out the calculation of $W(y_1, y_2)$ with $y_1 = e^{\lambda t}\cos(\mu t)$ and $y_2 = e^{\lambda t}\sin(\mu t)$.

Solution: We have

$$\begin{aligned} W(e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)) &= \begin{vmatrix} e^{\lambda t}\cos(\mu t) & \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) \\ e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix} \\ &= \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t) \\ &\quad -\lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t) \\ &= \mu e^{2\lambda t}(\sin^2(\mu t) + \cos^2(\mu t)) \\ &= \mu e^{2\lambda t}. \end{aligned}$$

As long as $\mu \neq 0$, this expression is non-zero, so the two solutions form a generating set.

C1. Solve the initial value problem $y'' + y' - 2y = t^2 + e^{3t}; \quad y(0) = 1,\ y'(0) = 2.$

The characteristic equation is $r^2 + r - 2 = 0$, which factors as $(r-1)(r+2) = 0$. The complementary solution to this differential equation is therefore

$$y_c = c_1 e^t + c_2 e^{-2t}.$$

The particular solution will take the form

$$Y = At^2 + Bt + C + De^{3t}.$$

From this we get

$$\begin{aligned} Y' &= 2At + B + 3De^{3t} \\ Y'' &= 2A + 9De^{3t}. \end{aligned}$$

Plugging these back into the given equation yields

$$\begin{aligned} t^2 + e^{3t} &= (2A + 9De^{3t}) + (2At + B + 3De^{3t}) - 2(At^2 + Bt + C + De^{3t}) \\ &= -4At^2 + (2A - 2B)t + (2A + B - 2C) + (9D + 3D - 2D)e^{3t}. \end{aligned}$$

We equate coefficients to get the system

$$\begin{array}{rcrcrcl} -2A & & & & & = & 1 \\ 2A & - & 2B & & & = & 0 \\ 2A & + & B & - & 2C & = & 0 \\ 10D & = & 1 & & & & \end{array}$$

The solution to the system is $A = B = -\dfrac{1}{2}$, $C = -\dfrac{3}{4}$ and $D = \dfrac{1}{10}$. The general solution to the differential equation is

$$y = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4} + \frac{1}{10}e^{3t} + c_1 e^t + c_2 e^{-2t}.$$

From this we get

$$y' = -t - \frac{1}{2} + \frac{3}{10}e^{3t} + c_1 e^t - 2c_2 e^{-2t}.$$

The initial conditions give us the system

$$\begin{aligned} -\frac{3}{4} + \frac{1}{10} + c_1 + c_2 &= 1 \\ -\frac{1}{2} + \frac{3}{10} + c_1 - 2c_2 &= 2. \end{aligned}$$

The solution is $c_1 = \frac{11}{6}$, $c_2 = -\frac{11}{60}$. The soluiton to the IVP is

$$y = \frac{11}{6}e^t - \frac{11}{60}e^{-2t} - \frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4} + \frac{1}{10}e^{3t}.$$

C2. Recall that if $y(t)$ gives the position of an object at time t, then $y''(t)$ is the object's acceleration. Also recall Newton's law: the acceleration of an object is a constant (namely, $1/m$) times the force acting on the object. The initial value problem

$$y'' = \cos t - \beta y'; \quad y(0) = 0, \; y'(0) = 0 \tag{2}$$

thus models an object (with mass 1) moving under the influence of an outside force $\cos t$ and a second force $-\beta y'$ that pushes against the object's motion, with a magnitude proportional to the object's velocity (we assume $\beta > 0$). The second force is a reasonable way to model sliding friction. The parameter β is the coefficient of friction.

(a) Solve the initial value problem (2). All the constants in your solution will depend on β

Solution: The differential equation is $y'' + \beta y' = \cos t$. The characteristic equation is $r^2 + \beta r = 0$, which has roots 0 and $-\beta$. The complementary solution to the differential equation is thus

$$y_c = c_1 + c_2 e^{-\beta t}.$$

The particular solution has the form $Y = A\cos t + B\sin t$. From this we get

$$\begin{aligned} Y' &= -A\sin t + B\cos t \\ Y'' &= -A\cos t - B\sin t. \end{aligned}$$

Plugging these back into the differential equation, we get

$$\begin{aligned} \cos t &= -A\cos t - B\sin t - \beta A\sin t + \beta B\cos t \\ &= (-A + \beta B)\cos t - (B + \beta A)\sin t. \end{aligned}$$

Equating coefficients, we get the system

$$\begin{aligned} \beta B - A &= 1 \\ B + \beta A &= 0. \end{aligned}$$

The solution is $A = -\frac{1}{\beta^2+1}$, $B = \frac{\beta}{\beta^2+1}$. The general solution to the differential equation is

$$y = c_1 + c_2 e^{-\beta t} + \frac{1}{\beta^2+1}(\beta \sin t - \cos t).$$

From the initial condition $y(0) = 0$ we get

$$0 = c_1 + c_2 - \frac{1}{\beta^2+1}$$

We also find that

$$y' = -\beta c_2 e^{-\beta t} + \frac{1}{\beta^2+1}(\beta \cos t + \sin t).$$

From the initial condition $y'(0) = 0$, we get

$$0 = -\beta c_2 + \frac{\beta}{\beta^2+1}.$$

We can now determine c_1 and c_2. We have

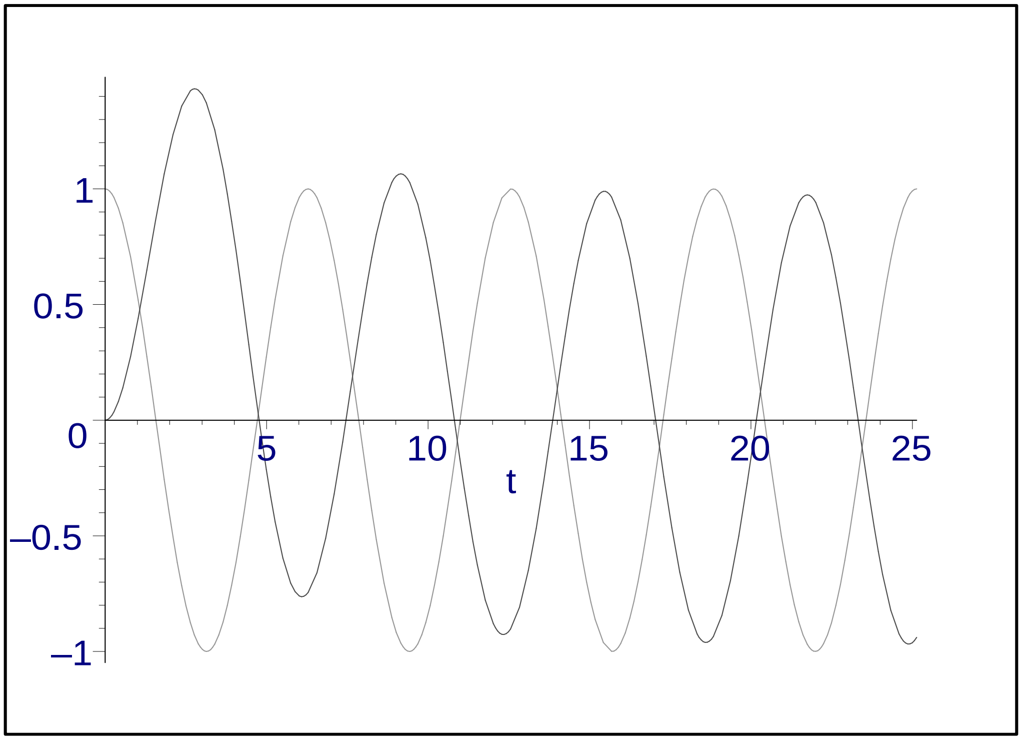
$$c_2 = \frac{1}{\beta^2+1}$$

from which it follows that $c_1 = 0$. The solution to the IVP is

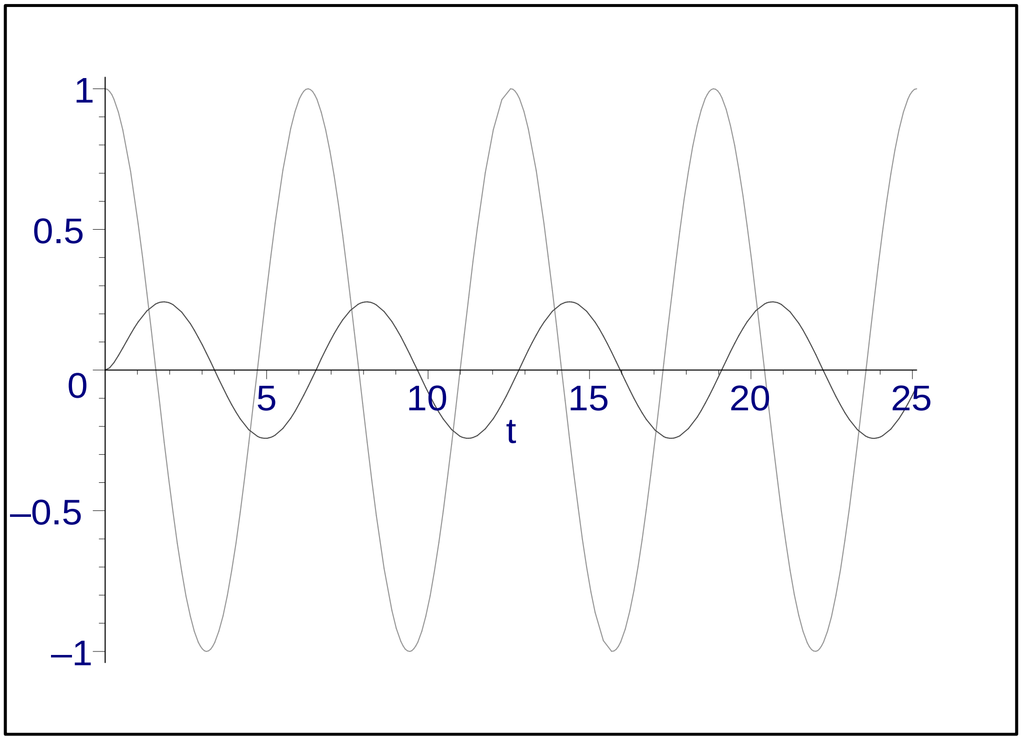
$$y = \frac{1}{\beta^2+1}(e^{-\beta t} + \beta \sin t - \cos t)$$

(b) Use the computer to plot solutions to (2) with $\beta = 0.25$, $\beta = 4$, and $\beta = 8$. With each solution plot, also plot the function $\cos t$ (the forcing function).

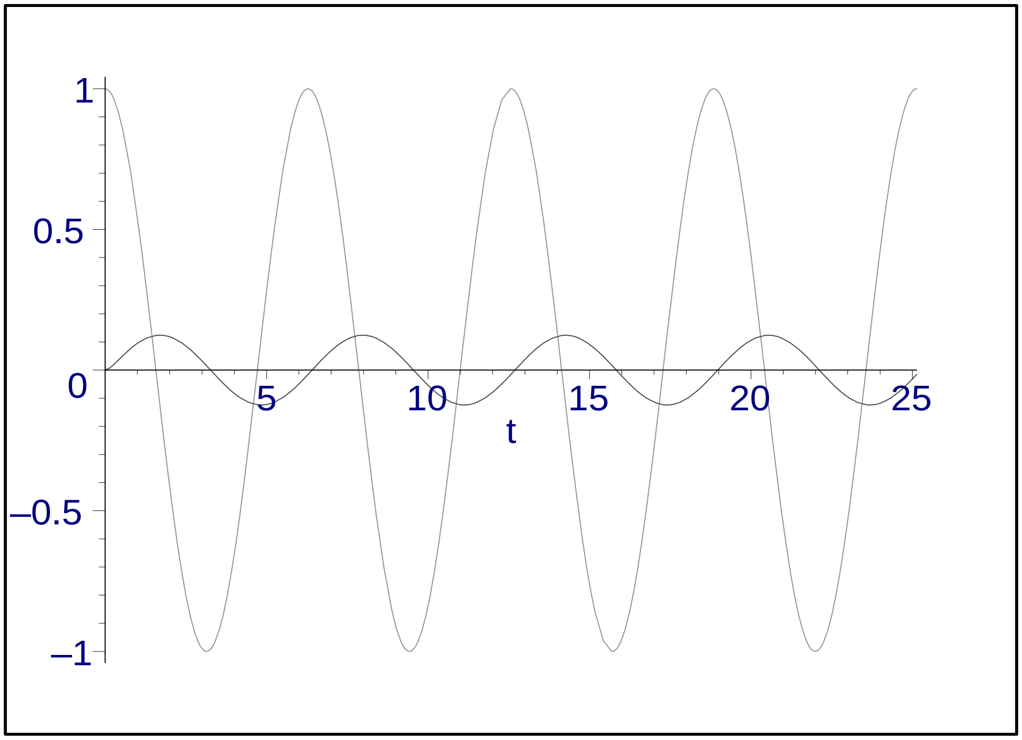
Here are the pictures: With $\beta = 0.25$:



With $\beta = 4$:



With $\beta = 8$:



(c) Comment on the relation between the forcing function and the response function (that is, the solution) for the various values of β. How do the amplitude, frequency, and phase shift of the response seem to depend on β?

The amplitude of the response decreases with increasing β. One would expect this, since β is a drag coefficient.

(More precisely, the amplitude of the steady-state solution is $\dfrac{\sqrt{\beta^2+1}}{\beta^2+1}$.)

The frequency of the response is equal in all cases to the frequency of the driving function.

The phase shift of the response seems to depend on β in the following way: For small values of β, the response lags approximately 180° behind the driving function. For larger values of β, the phase lag decreases. When $\beta = 8$, the response seems to lag about 90° behind the driving function.

(More precisely, we have a phase lag φ determined by $\tan\varphi = \beta$, with $\varphi = -\pi$ when $\beta = 0$. Thus the phase lag φ is given by $-\pi + \tan^{-1}\beta$, so it does indeed approach $-\dfrac{\pi}{2}$ as $\beta \to \infty$.)