A1. (a) Solve the initial value problem

$$y'' + 2y' + 10y = 37\cos 3t; \quad y(0) = 0, \quad y'(0) = 1.$$

Identify which terms in your solution belong to the transient solution and which belong to the steady-state solution.

Solution: The characteristic equation is $r^2 + 2r + 10 = 0$; the roots are $-1 \pm 3i$, so the complementary solution has the form

$$y_c = e^{-t}(c_1 \cos 3t + c_2 \sin 3t).$$

The particular solution has the form

$$Y = A\cos 3t + B\sin 3t.$$

From this we get

$$\begin{aligned} Y' &= -3A\sin 3t + 3B\cos 3t \\ Y'' &= -9A\cos 3t - 9B\cos 3t. \end{aligned}$$

Making the substitutions in the original DE gives

$$(-9A + 6B + 10A)\cos 3t + (-9B - 6A + 10B)\sin 3t = 37\cos 3t.$$

This gives us the system

$$\begin{aligned} A + 6B &= 37 \\ -6A + B &= 0 \end{aligned}$$

The solution is $A = 1,\ B = 6$.
The solution to the initial value problem now reads

$$y = \cos 3t + 6\sin 3t + e^{-t}(c_1\cos 3t + c_2 \sin 3t).$$

from this we get

$$\begin{aligned} y' = & -3\sin 3t + 18\cos 3t - e^{-t}(c_1\cos 3t + c_2\sin 3t) + \\ & e^{-t}(-3c_1\sin 3t + 3c_2\cos 3t). \end{aligned}$$

The initial conditions imply that

$$0 = y(0) = 1 + c_1$$

so that $c_1 = -1$, and

$$1 = y'(0) = 18 - c_1 + 3c_2$$

so that $c_2 = -6$.

The complete solution is

$$y(t) \quad = \quad \cos 3t + 6 \sin 3t - e^{-t}(\cos 3t + 6 \sin 3t).$$

The first two terms are the steady-state solution. The terms multiplied by e^{-t} are the transient solution.

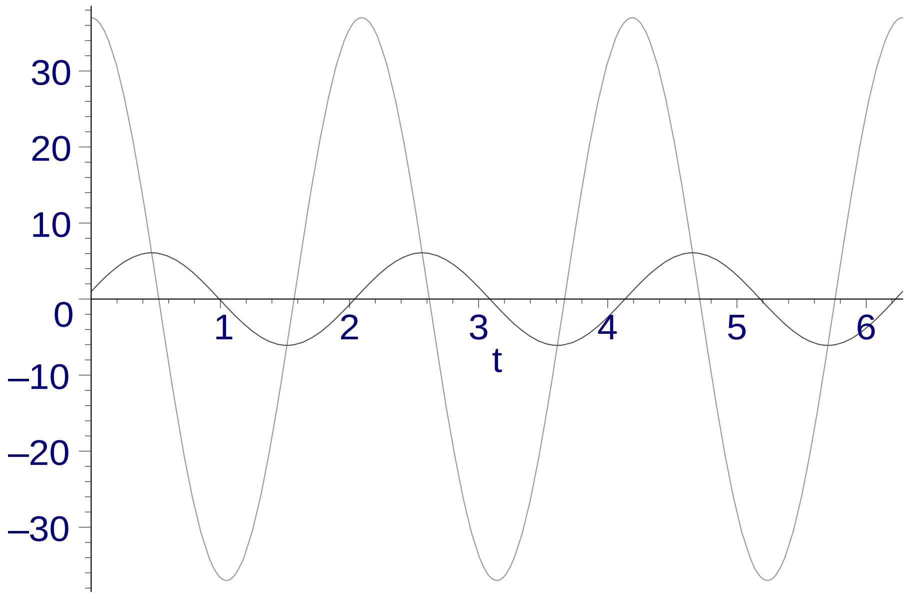
(b) Compare the amplitude and phase angle of the steady-state solution with those of the function $37 \cos 3t$.

Solution: The steady-state solution above can be written as

$$\sqrt{37} \cos(3t - \varphi)$$

where $\cos \varphi = \dfrac{1}{\sqrt{37}}$ and $\sin \varphi = \dfrac{6}{\sqrt{37}}$. The angle φ with these trig values is approximately 1.4056 radians, or about 80.5 degrees. The steady-state response has a smaller amplitude than the forcing function (by a factor of $\sqrt{37}$) and lags about 80.5 degrees behind the forcing function.

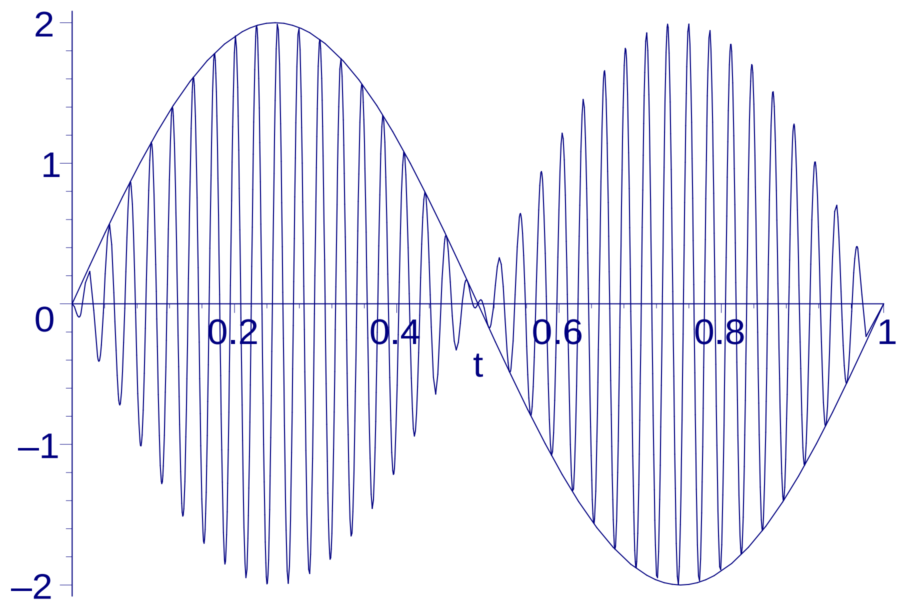
Here is a picture showing the driving function (with the larger amplitude) and the response function:



A2. Consider a mass and spring system with no damping, driven by a sinusoid forcing function at 500 cycles per second (nb. *cycles*, not radians). In the response, we observe beats. The amplitude of the response function rises and falls two times per second.

Suppose $m = 100$ grams. Find the possible values for k, the spring constant.

Solution: Solution: Let ω_0 denote the natural frequency of the system in radians per second. The system is driven at 500 cycles per second, which is the same as 1000π radians per second.



As the picture above suggests, we hear two amplitude peaks per second when the beat frequency is 2π radians per second.

This tells us that

$$\frac{|\omega_0 - 1000\pi|}{2} \quad = \quad 2\pi \text{ radians per second.}$$

Thus we have

$$\omega_0 \quad = \quad 1000\pi \pm 4\pi \text{ radians per second}$$

so that we have

$$\begin{aligned} \frac{\sqrt{k}}{10} &= (1000 \pm 4)\pi \\ \sqrt{k} &= (10000 \pm 40)\pi \\ k &= (10000 \pm 40)^2\pi^2 \text{ dynes per centimeter.} \end{aligned}$$

B1. (B & D, §3.9, problem 17) Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 2\cos\omega t, \quad u(0) = 0, \quad u'(0) = 2.$$

(a) Determine the steady-state part of the solution of this problem.

Solution: Following the notation on p. 203 of B&D, we have $m = 1$, $k = 2$, so $\omega_0 = \sqrt{2}$, and $\gamma = 1/4$. From this we get

$$\Delta = \sqrt{(2-\omega^2)^2 + \omega^2/16}$$

With $F_0 = 2$, we can write the steady-state solution as

$$U(t) = \frac{2}{\sqrt{(2-\omega^2)^2 + \omega^2/16}}\cos(\omega t - \delta)$$

where

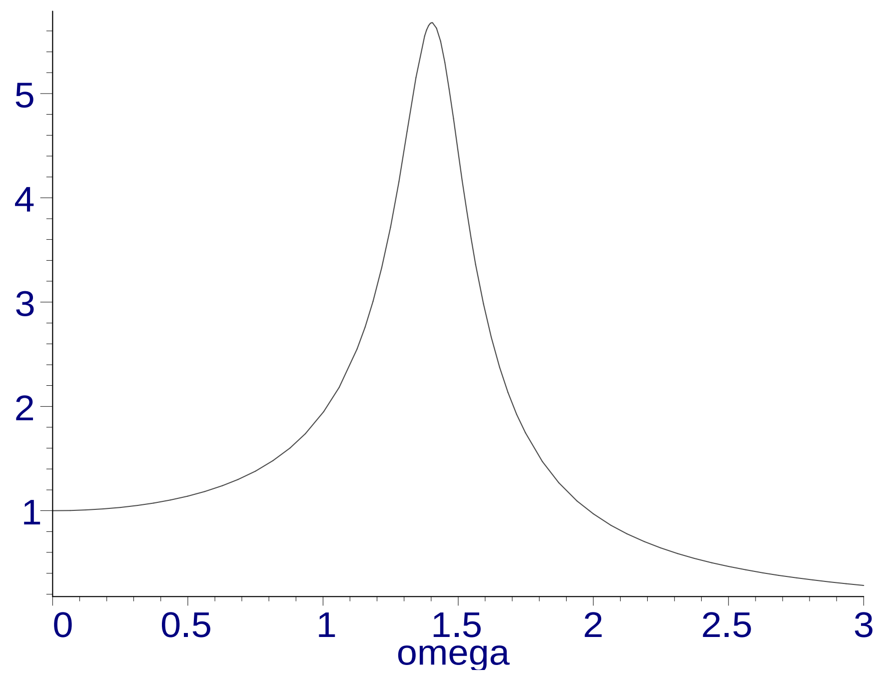
$$\cos\delta = \frac{2-\omega^2}{\sqrt{(2-\omega^2)^2 + \omega^2/16}} \qquad \sin\delta = \frac{\omega}{4\sqrt{(2-\omega^2)^2 + \omega^2/16}}$$

(b) Find the amplitude A of the steady-state solution in terms of ω.

Solution: We have done this already; it's $\dfrac{2}{\sqrt{(2-\omega^2)^2 + \omega^2/16}}$.

(c) Plot A as a function of ω.

Solution: Here is the picture:



(d) Find the maximum value of A and the frequency ω for which it occurs.

Solution: Again following B&D p. 203, we have

$$
\begin{aligned}
R_{\max} &= \frac{2}{(\sqrt{2}/4)\sqrt{1-(1/128)}} \\
&= \frac{8}{\sqrt{2-\frac{1}{64}}} \\
&\approx 5.679.
\end{aligned}
$$

The value of ω that gives this peak is

$$
\begin{aligned}
\omega_{\max} &= \sqrt{2-\frac{1}{32}} \\
&= \sqrt{\frac{63}{32}} \\
&= 1.403.
\end{aligned}
$$

B2. Consider a series circuit with a 10 microfarad capacitor (a microfarad is 10^{-6} farad), a 0.25 henry inductor, and a 10 ohm resistor.

(a) Assume the impressed voltage is 0, the initial charge on the capacitor is 10^{-5} coulomb, and the initial current is 0.

Find $Q(t)$, the charge on the capacitor, as a function of time.

Solution: The initial value problem here is

$$\frac{1}{4}Q'' + 10Q' + 10^5 Q = 0; \quad Q(0) = 10^{-5}, \quad Q'(0) = 0.$$

The characteristic equation is $\frac{1}{4}r^2 + 10r + 10^5 = 0$. The roots are $-20 \pm 60i\sqrt{111}$. The general solution to this differential equation is

$$Q(t) = e^{-20t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

where $\omega = 60\sqrt{111}$. From this we get

$$Q'(t) = -20e^{-20t}(c_1 \cos \omega t + c_2 \sin \omega t) + e^{-20t}(-\omega c_1 \sin \omega t + \omega c_2 \cos \omega t).$$

The initial conditions say

$$10^{-5} = Q(0) = c_1$$

and

$$0 = Q'(0) = -20c_1 + \omega c_2$$

Thus $c_1 = 10^{-5}$ and $c_2 = \frac{20c_1}{\omega} = \frac{10^{-5}}{3\sqrt{111}}$. The complete solution is

$$Q(t) = 10^{-5}e^{-20t}\left(\cos \omega t + \frac{1}{3\sqrt{111}} \sin \omega t\right).$$

(b) Now assume an impressed voltage (varying with time) given by

$$E(t) = 20\cos\omega t.$$

For what value of ω will the circuit's steady-state response be the greatest? What is the maximum voltage drop across the capacitor when this value of ω is used? Give exact values and decimal approximations.

Solution: Once again we go to p. 203. For this problem, we find that $\omega_0 = \sqrt{4 \times 10^5} = 400\sqrt{10}$. With $\gamma = 10$, we get

$$\begin{aligned} \omega_{\max}^2 &= 4 \times 10^5 - \frac{10^2}{2/16} \\ &= 4 \times 10^5 - 800 \end{aligned}$$

so that the maximum response occurs when

$$\begin{aligned} \omega &= \sqrt{4 \times 10^5 - 800} \\ &\approx 631.823 \text{ radians per second.} \end{aligned}$$

With this value of ω, the amplitude of the response is

$$\begin{aligned} R_{\max} &= \frac{20}{10\sqrt{4 \times 10^5}\sqrt{1 - \frac{10^2}{10^5}}} \\ &= \frac{1}{30\sqrt{111}}. \end{aligned}$$

The maximum voltage drop across the capacitor is $1/C$ times this value. That is,

$$\frac{10^5}{20\sqrt{111}} \approx 316.386 \text{ volts.}$$

C1. Use variation of parameters to find the general solution to the differential equation

$$y'' + 4y = \sec 2t.$$

Solution: The characteristic equation of the corresponding homogeneous differential equation is $r^2 + 4 = 0$. The roots of this equation are $\pm 2i$, so the general solution to the homogeneous equation is

$$y_c = c_1 \cos 2t + c_2 \sin 2t.$$

We write

$$y = u_1(t) \cos 2t + u_2(t) \sin 2t$$

and differentiate to get

$$y' = u_1' \cos 2t - 2u_1 \sin 2t + u_2' \sin 2t + 2u_2 \cos 2t.$$

Now we impose the condition

$$u_1' \cos 2t + u_2' \sin 2t = 0 \tag{1}$$

so that $y' = -2u_1 \sin 2t + 2u_2 \cos 2t$. From this we get

$$y'' = -2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t.$$

Substituting our expressions for y'' and y back into the original differential equation yields

$$\begin{aligned} \sec 2t &= y'' + 4y \\ &= -2u_1' \sin 2t - 4u_1 \cos 2t + 2u_2' \cos 2t - 4u_2 \sin 2t \\ &\quad\ +4u_1 \cos 2t + 4u_2 \sin 2t \\ &= -2u_1' \sin 2t + 2u_2' \sin 2t. \end{aligned}$$

Combining this with equation (1), we get the system

$$u_1' \cos 2t + u_2' \sin 2t = 0 \tag{2}$$
$$-2u_1' \sin 2t + 2u_2' \sin 2t = \sec 2t. \tag{3}$$

Multiplying the top equation by $2 \sin 2t$ and the bottom by $2 \cos 2t$ gives

$$\begin{aligned} 2u_1' \cos 2t \sin 2t + 2u_2' \sin^2 2t &= 0 \\ -2u_1' \cos 2t \sin 2t + 2u_2' \cos^2 2t &= 1. \end{aligned}$$

Adding these two equations, we get $2u_2' = 1$, from which $u_2' = \dfrac{1}{2}$. Substituting this value for u_2' back into equation (2) gives

$$u_1' \cos 2t + \frac{\sin 2t}{2} \quad = \quad 0$$

which we solve to get

$$u_1' \quad = \quad -\frac{\tan 2t}{2}.$$

Integrating u_1' and u_2', we get

$$\begin{aligned} u_1 &= -\frac{\ln(\sec 2t)}{4} + c_1 \\ u_2 &= \frac{t}{2} + c_2. \end{aligned}$$

Substituting these functions back into our original solution, we get the general solution

$$y \quad = \quad c_1 \cos 2t - \frac{\ln(\sec 2t)}{4} \cos 2t + c_2 \sin 2t + \frac{t \sin 2t}{2}.$$

C2. (B & D, §3.7, problem 17) Verify that $y_1 = x^2$ and $y_2 = x^2 \ln x$ (with $x > 0$) are solutions to the differential equation

$$x^2 y'' - 3xy' + 4y = 0.$$

Find a particular solution the differential equation $x^2 y'' - 3xy' + 4y = x^2 \ln x$.

Solution: Given the differential equation

$$x^2 y'' - 3xy' + 4y \quad = \quad x^2 \ln x,$$

we first check that the given functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are indeed solutions to the corresponding homogeneous equation.

We have

$$\begin{aligned} y_1 &= x^2 \\ y_1' &= 2x \\ y_1'' &= 2, \end{aligned}$$

so that

$$\begin{aligned} x^2y_1'' - 3xy_1' + 4y_1 &= x^2(2) - 3x(2x) + 4x^2 \\ &= 2x^2 - 6x^2 + 4x^2 \\ &= 0. \end{aligned}$$

This shows that y_1 is a solution to the corresponding homogeneous equation. Similarly, we have

$$\begin{aligned} y_2 &= x^2 \ln x \\ y_2' &= 2x \ln x + x \\ y_2'' &= 2 \ln x + 3, \end{aligned}$$

so that

$$\begin{aligned} x^2y_2'' - 3xy_2' + 4y_2 &= x^2(2\ln x + 3) - 3x(2x \ln x + x) + 4(x^2 \ln x) \\ &= (2x^2 - 6x^2 + 4x^2)\ln x + 3x^2 - 3x^2 \\ &= 0. \end{aligned}$$

To solve the non-homogeneous differential equation, we use variation of parameters. We let

$$y = u_1x^2 + u_2x^2 \ln x. \tag{4}$$

Then we have

$$\begin{aligned} y' &= u_1'x^2 + 2xu_1 + u_2'x^2 \ln x + (2x \ln x + x)u_2 \\ &= 2xu_1 + (2x \ln x + x)u_2 + x^2u_1' + (x^2 \ln x)u_2'. \end{aligned}$$

We impose the condition

$$x^2u_1' + (x^2 \ln x)u_2' = 0 \tag{5}$$

so that we have

$$y' = 2xu_1 + (2x \ln x + x)u_2,$$

from which it follows that

$$y'' = 2u_1 + 2xu_1' + (2\ln x + 3)u_2 + (2x \ln x + x)u_2'.$$

Substituting our expressions for y, y', and y'' into the original differential equation, we get

$$\begin{aligned}
x^2 \ln x &= x^2(2u_1 + 2xu_1' + (2\ln x + 3)u_2 + (2x\ln x + x)u_2') \\
&\quad -3x(2xu_1 + (2x\ln x + x)u_2) + 4(x^2 u_1 + (x^2 \ln x)u_2) \\
&= 2x^3 u_1' + (2x^3 \ln x + x^3)u_2' \\
&\quad +(2x^2 - 6x^2 + 4x^2)u_1 + (2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x)u_2 \\
&= 2x^3 u_1' + (2x^3 \ln x + x^3)u_2'.
\end{aligned}$$

The coefficients of u_1 and u_2 are both zero, as expected.

Next we recall condition (5) and solve the system

$$\begin{aligned}
x^2 u_1' + (x^2 \ln x)u_2' &= 0 \\
2x^3 u_1' + (2x^3 \ln x + x^3)u_2' &= x^2 \ln x.
\end{aligned}$$

Since we are assuming $x > 0$, we may divide through by suitable powers of x to get the somewhat simpler system

$$u_1' + (\ln x)u_2' = 0 \tag{6}$$

$$2u_1' + (2\ln x + 1)u_2' = \frac{\ln x}{x}. \tag{7}$$

From (6), we get $u_1' = -(\ln x)u_2'$. Making this substitution in (7) yields

$$-2\ln x u_2' + 2\ln x u_2' + u_2' = \frac{\ln x}{x}$$

so that $u_2' = \dfrac{\ln x}{x}$. From this we get

$$\begin{aligned}
u_1' &= -(\ln x)u_2' \\
&= -\frac{(\ln x)^2}{x}.
\end{aligned}$$

We integrate u_1' to get

$$u_1 = -\frac{(\ln x)^3}{3} + c_1$$

and u_2' to get

$$u_2 = \frac{(\ln x)^2}{2} + c_2.$$

We plug u_1 and u_2 into expression (4) to get

$$y = -\frac{x^2(\ln x)^3}{3} + c_1x^2 + \frac{x^2(\ln x)^3}{2} + c_2x^2 \ln x.$$

Since $c_1x^2 + c^2x^2 \ln x$ is the complementary solution in this problem, the particular solution we seek is

$$Y(x) = \frac{x^2(\ln x)^3}{6}.$$