Math 333 — Problem Set #8 — Due 25 April 2003, 12:00 NOON

Reading: B & D, §§ 6.1 – 6.3

Exercises: Write your solutions clearly, and in complete sentences, remembering that they will be graded for presentation as well as correctness.

A1. Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t < 4 \\ 2 & \text{if } 4 \le t. \end{cases}$

A2. (a) Use integration by parts to prove the following:

Theorem: For $n \ge 1$ and $s > 0$, $\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$.

Be sure to point out where you use the hypothesis $s > 0$.

(b) Given that $\mathcal{L}\{1\} = \frac{1}{s}$ (for $s > 0$), use the formula in part A2a to find $\mathcal{L}\{t\}$, $\mathcal{L}\{t^2\}$, and $\mathcal{L}\{t^3\}$. Based on these results, make a conjecture about the value of $\mathcal{L}\{t^n\}$ for any positive integer n.

(Optionally, use induction to prove that your conjecture is correct.)

B1. Use Laplace transforms to solve the initial value problem

$$y'' + 5y' + 6y = g(t), \quad y(0) = y'(0) = 0,$$

where

$$g(t) = \begin{cases} 6 & \text{if } 0 \le t < 1 \\ -6 & \text{if } 1 \le t < 2 \\ 0 & \text{if } 2 \le t. \end{cases}$$

Then use a computer to plot your solution.

B2. (a) Line 19 of the Laplace transform table says that if $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

Use this fact to show that

$$\mathcal{L}\{t\cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

(b) Use the result from part (B2a) and the Laplace transform table to show that

$$\mathcal{L}\left\{\frac{1}{a}\sin at - t\cos at\right\} = \frac{2a^2}{(s^2+a^2)^2}.$$

(c) Use Laplace transforms to solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = y'(0) = 0$$

where

$$g(t) = \begin{cases} \sin 2t & \text{if } 0 \le t < 4\pi \\ 0 & \text{if } 4\pi \le t. \end{cases}$$

Use a computer to plot your solution.

C1. Let g be the square wave given (for $t \ge 0$) by

$$g(t) = \sum_{n=0}^{\infty} (-1)^n u_n(t)$$

(a) Graph $g(t)$, either by hand or with the computer. In Maple, the step function is called `Heaviside()`, and the translation is

$$u_c(t) = \texttt{Heaviside(t} - \texttt{c)}$$

(b) Let $G(s)$ be the Laplace transform of $g(t)$. Find the simplest expression you can for $G(s)$. Don't forget to indicate the set of s for which $G(s)$ converges.

You may want to use the fact that $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

C2. Solve the initial value problem $y'' + 8y = g(t)$; $\quad y(0) = 0, \quad y'(0) = 0$ where $g(t)$ is the square wave in problem C1. Plot your solution for t from 0 to at least 50.

(Note: it will be easier to leave the transform of $g(t)$ in the form of an infinite sum.)