A1. Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 0 & \text{if } 0 \le t < 2 \\ t-2 & \text{if } 2 \le t < 4 \\ 2 & \text{if } 4 \le t. \end{cases}$

Solution: Writing g in terms of Heaviside functions, we get

$$\begin{aligned} g(t) &= (t-2)(u_2(t) - u_4(t)) + 2u_4(t) \\ &= (t-2)(u_2(t)) - (t-4)u_4(t). \end{aligned}$$

Since the Laplace transform of $(t-c)u_c(t)$ is e^{-cs} times the Laplace transform of t, we get

$$\mathcal{L}\{g(t)\} = \frac{e^{-2s}}{s^2} - \frac{e^{-4s}}{s^2}.$$

A2. (a) Use integration by parts to prove the following:

Theorem: For $n \ge 1$ and $s > 0$, $\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$.

Be sure to point out where you use the hypothesis $s > 0$.

Solution: By definition of the Laplace transform,

$$\begin{aligned} \mathcal{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt. \\ &= \lim_{A\to\infty} \int_0^A e^{-st} t^n \, dt. \end{aligned}$$

We let $u = t^n \, dt$ and $v = e^{-st} \, dt$ and apply integration by parts. We have $du = nt^{n-1} \, dt$ and, since $s \ne 0$, $v = -\frac{e^{-st}}{s}$.

$$\begin{aligned} \lim_{A\to\infty} \int_0^A e^{-st} t^n \, dt &= \lim_{A\to\infty} \left[-\frac{t^n e^{-st}}{s} \right]_0^A + \int_0^\infty nt^{n-1} \frac{e^{-st}}{s} \, dt \\ &= \lim_{A\to\infty} \left[0 + \frac{A^n e^{-sA}}{s} \right] + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt. \end{aligned}$$

By the definition of the Laplace transform, the second term on the right is $\frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}$. The limit in the first term on the right is

$$\frac{1}{s}\lim_{A\to\infty}\frac{A^n}{e^{sA}}.$$

Since $s > 0$, we have an increasing exponential in the denominator, and a polynomial in the numerator. Since exponentials grow faster than polynomials (see below), we have $\lim_{A\to\infty}\frac{A^n}{e^{sA}} = 0$, and thus the first term on the right is 0. In summary, we have

$$\mathcal{L}\left\{t^n\right\} = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}.$$

for $n \geq 1$ and $s > 0$.

Here is a proof that $\lim_{x\to\infty}\frac{x^n}{e^{ax}} = 0$ for $a > 0$ and $n \geq 0$. We proceed by induction on n.

<u>Base case: $n = 0$</u>. We have $\lim_{x\to\infty}\frac{1}{e^{ax}}$, which is certainly zero, since $e^{ax} \to \infty$ as $x \to \infty$.

<u>Inductive step</u>. Let $n \geq 1$, and assume that $\lim_{x\to\infty}\frac{x^{n-1}}{e^{ax}} = 0$. Since $n \geq 1$, we know that $x^n \to \infty$ as $x \to \infty$, and since $a > 0$, we know that $e^{ax} \to \infty$ as well. We may apply l'Hospital's rule to get

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^n}{e^{ax}} &= \lim_{x\to\infty}\frac{nx^{n-1}}{ae^{ax}} \\
&= \frac{n}{a}\lim_{x\to\infty}\frac{x^{n-1}}{e^{ax}} \\
&= \frac{n}{a}\cdot 0. \\
&= 0.
\end{aligned}$$

(b) Given that $\mathcal{L}\left\{1\right\} = \frac{1}{s}$ (for $s > 0$), use the formula in part A2a to find $\mathcal{L}\left\{t\right\}$, $\mathcal{L}\left\{t^2\right\}$, and $\mathcal{L}\left\{t^3\right\}$. Based on these results, make a conjecture about the value of $\mathcal{L}\left\{t^n\right\}$ for any positive integer n.

(Optionally, use induction to prove that your conjecture is correct.)

Solution: We have

$$\begin{aligned}
\mathcal{L}\{t\} &= \frac{1}{s}\mathcal{L}\{1\} \\
&= \frac{1}{s^2} \quad (s>0) \\
\mathcal{L}\left\{t^2\right\} &= \frac{2}{s}\mathcal{L}\{t\} \\
&= \frac{2}{s^3} \quad (s>0) \\
\mathcal{L}\left\{t^3\right\} &= \frac{3}{s}\mathcal{L}\left\{t^2\right\} \\
&= \frac{6}{s^4} \quad (s>0).
\end{aligned}$$

Each time we increase the exponent on t, we multiply the denominator of the transform by s and the numerator by the new exponent. We conjecture that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (s>0).$$

Here's an inductive proof.

We are given that $\mathcal{L}\left\{t^0\right\} = \frac{1}{s}$, so the conjecture holds for $n=0$.

Now suppose that $n \geq 1$ and $\mathcal{L}\left\{t^{n-1}\right\} = \frac{(n-1)!}{s^n}$ for $s>0$. We want to show that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ for } s>0.$$

By the result above, we have

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\} \\
&= \frac{n}{s}\frac{(n-1)!}{s^n} \\
&= \frac{n!}{s^{n+1}},
\end{aligned}$$

valid for $s>0$. This completes the proof.

B1. Use Laplace transforms to solve the initial value problem

$$y'' + 5y' + 6y = g(t), \quad y(0) = y'(0) = 0,$$

where

$$g(t) = \begin{cases} 6 & \text{if } 0 \le t < 1 \\ -6 & \text{if } 1 \le t < 2 \\ 0 & \text{if } 2 \le t. \end{cases}$$

Then use a computer to plot your solution.

Solution: Writing $g(t)$ in terms of Heaviside functions, we get

$$g(t) = 6(1 - 2u_1(t) + u_2(t)).$$

Taking the Laplace transform of both sides of the given equation yields

$$(s^2 + 5s + 6)Y = \frac{6}{s}(1 - 2e^{-t} + e^{-2t})$$

where Y is the Laplace transform of our solution. Solving for Y, we get

$$Y = \frac{6}{s(s^2 + 5s + 6)}(1 - 2e^{-t} + e^{-2t}).$$

It will be convenient to set

$$H(s) = \frac{6}{s(s^2 + 5s + 6)}$$

and let $h(t)$ denote the inverse Laplace transform of $H(s)$.

The denominator of $H(s)$ factors as $s(s+2)(s+3)$, and using partial fractions, we find that

$$H(s) = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

so that

$$h(t) = 1 - 3e^{-2t} + 2e^{-3t}. \tag{1}$$

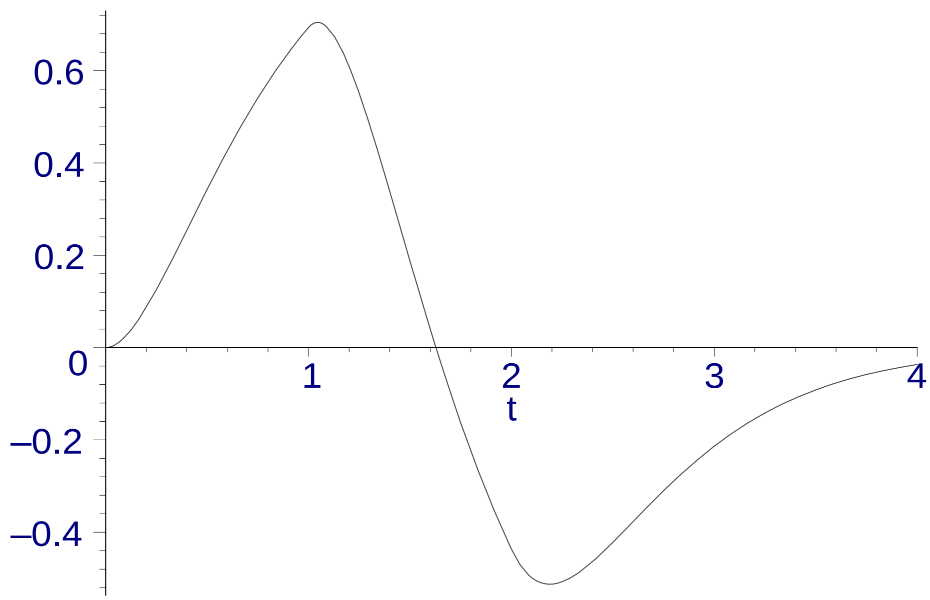
It follows that our solution $y(t)$, the inverse Laplace transform of $Y(s)$ is given by

$$y(t) = h(t) - 2u_1(t)h(t-1) + u_2(t)h(t-2)$$

where $h(t)$ is given in equation (1).

Here is the solution, plotted by *Maple*:

```
> hb := t -> 1 - 3*exp(-2*t) + 2*exp(-3*t);
> yb := t -> hb(t) - 2*Heaviside(t-1)*hb(t-1) +
> Heaviside(t-2)*hb(t-2);
> plot(yb(t),t=0..4);
```



B2. (a) Line 19 of the Laplace transform table says that if $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{tf(t)\} \;=\; -F'(s).$$

Use this fact to show that

$$\mathcal{L}\{t\cos at\} \;=\; \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Solution: Given $f(t) = \cos at$ and $F(s) = \dfrac{s}{s^2+a^2}$, we have

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -F'(s)\\ &= -\frac{(s^2+a^2)-2s^2}{(s^2+a^2)^2}\\ &= \frac{s^2-a^2}{(s^2+a^2)^2},\end{aligned}$$

as required.

(b) Use the result from part (B2a) and the Laplace transform table to show that

$$\mathcal{L}\left\{\frac{1}{a}\sin at - t\cos at\right\} = \frac{2a^2}{(s^2+a^2)^2}.$$

Solution: We have

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{a}\sin at - t\cos at\right\} &= \frac{1}{a}\left(\frac{a}{s^2+a^2}\right) - \frac{s^2-a^2}{(s^2+a^2)^2}\\ &= \frac{1}{s^2+a^2} - \frac{s^2-a^2}{(s^2+a^2)^2}\\ &= \frac{s^2+a^2-(s^2-a^2)}{(s^2+a^2)^2}\\ &= \frac{2a^2}{(s^2+a^2)^2},\end{aligned}$$

as required.

(c) Use Laplace transforms to solve the initial value problem

$$y'' + 4y = g(t), \quad y(0) = y'(0) = 0$$

where

$$g(t) = \begin{cases} \sin 2t & \text{if } 0 \le t < 4\pi \\ 0 & \text{if } 4\pi \le t. \end{cases}$$

Use a computer to plot your solution.

Solution: Writing $g(t)$ in terms of Heaviside functions, we get

$$\begin{aligned}g(t) &= \sin 2t(1-u_{4\pi}(t))\\ &= \sin 2t - u_{4\pi}(t)\sin 2t\\ &= \sin 2t - u_{4\pi}(t)\sin(2(t-4\pi)),\end{aligned}$$

where the last equality follows because sine is obligingly 2π-periodic (and therefore 8π-periodic).

Taking the Laplace transforms of both sides of our equation, we get

$$(s^2+4)Y = (1-e^{-4\pi s})\frac{2}{s^2+4},$$

where Y is the Laplace transform of the solution. Solving for Y, we get

$$Y = (1-e^{-4\pi s})H(s) \tag{2}$$

where

$$\begin{aligned} H(s) &= \frac{2}{(s^2+4)^2} \\ &= \frac{1}{4}\left(\frac{2\times 2^2}{(s^2+2^2)^2}\right). \end{aligned}$$

From the results above, we know that the inverse transform $h(t)$ of $H(s)$ is given by

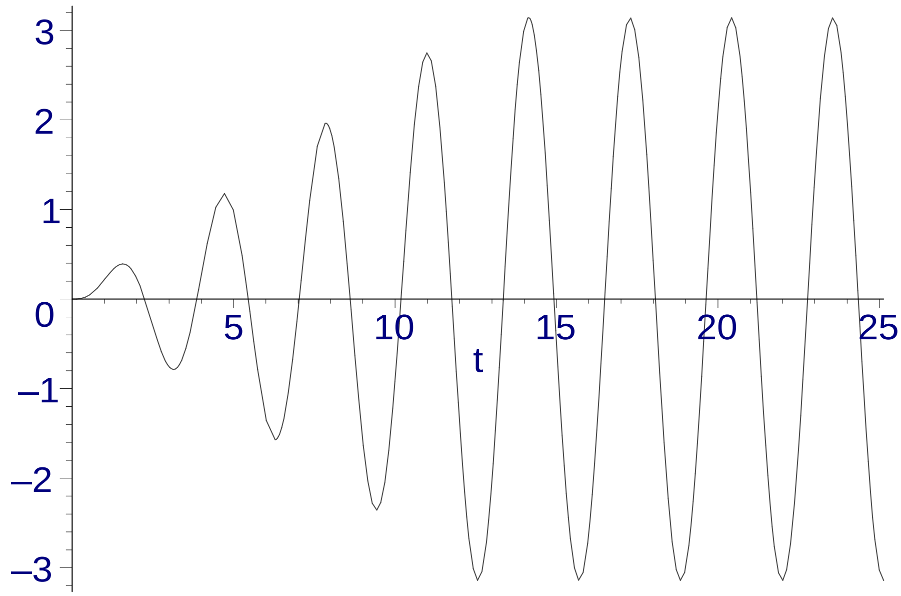
$$\begin{aligned} h(t) &= \frac{1}{4}\left(\frac{1}{2}\sin 2t - t\cos 2t\right) \\ &= \frac{\sin 2t}{8} - \frac{t\cos 2t}{4}. \end{aligned}$$

Then from equation (2) above, our solution y is given by

$$y(t) = h(t) - u_{4\pi}(t)h(t-4\pi).$$

Here is a picture

```
> hc := t -> (1/8)*sin(2*t) - (1/4)*t*cos(2*t);
> yc := t -> hc(t) - Heaviside(t-4*Pi)*hc(t-4*Pi);
> plot(yc(t),t=0..8*Pi);
```



C1. Let g be the square wave given (for $t \geq 0$) by

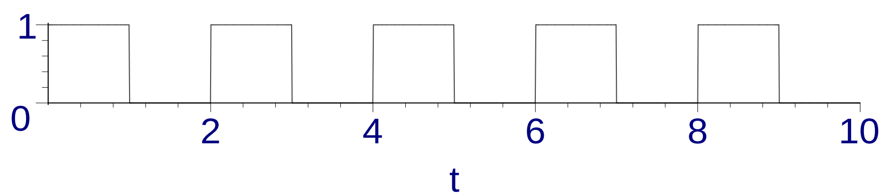
$$g(t) \;=\; \sum_{n=0}^{\infty} (-1)^n u_n(t)$$

(a) Graph $g(t)$, either by hand or with the computer. In Maple, the step function is called `Heaviside()`, and the translation is

$$u_c(t) \;=\; \texttt{Heaviside}(\texttt{t} - \texttt{c})$$

Solution: Here is a graph of $g(t)$. *Maple* has included vertical lines at all the points of discontinuity. It would.

```
> sq := (n,t) -> sum((-1)^k*Heaviside(t-k),k=0..n);
> plot(sq(10,t),t=0..10,scaling=CONSTRAINED);
```



(b) Let $G(s)$ be the Laplace transform of $g(t)$. Find the simplest expression you can for $G(s)$. Don't forget to indicate the set of s for which $G(s)$ converges.

You may want to use the fact that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$.

Solution: Since the Laplace transform is linear, we have

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \sum_{n=0}^{\infty} (-1)^n \mathcal{L}\{u_n(t)\} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{e^{-ns}}{s} \\
&= \frac{1}{s} \sum_{n=0}^{\infty} (-e^{-s})^n.
\end{aligned}$$

For $s > 0$, we have $|-e^{-s}| = e^{-s} < 1$, so the geometric series converges, and we

get

$$\mathcal{L}\{g(t)\} = \frac{1}{s} \cdot \frac{1}{1+e^{-s}}, \quad s > 0.$$

C2. Solve the initial value problem $y'' + 8y = g(t); \quad y(0) = 0, \quad y'(0) = 0$ where $g(t)$ is the square wave in problem C1. Plot your solution for t from 0 to at least 50.

(Note: it will be easier to leave the transform of $g(t)$ in the form of an infinite sum.)

Solution: Taking the transforms of both sides of the differential equation, we get

$$(s^2+8)Y = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

so that

$$Y = \frac{1}{s(s^2+8)}\sum_{n=0}^{\infty}(-1)^n e^{-ns}$$

Let $H(s) = \dfrac{1}{s(s^2+8)}$. By partial fractions, we find that

$$H(s) = \frac{1}{8}\left(\frac{1}{s} - \frac{s}{s^2+8}\right).$$

Let $h(t)$ be the inverse transform of $H(s)$. We have

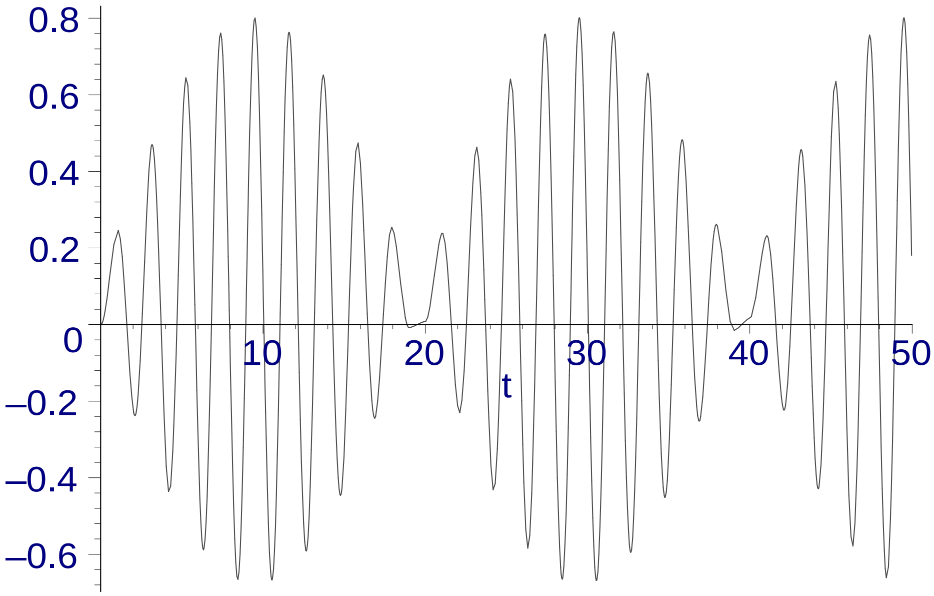
$$h(t) = \frac{1}{8} - \frac{1}{8}\cos(\sqrt{8}t).$$

The solution to the IVP is

$$\sum_{n=0}^{\infty}(-1)^n u_n(t) h(t-n).$$

Here is a plot of this solution.

```
> hd := t -> (1/8)*(1-cos(t*sqrt(8)));
> yd := (n,t) -> sum((-1)^k*Heaviside(t-k)*hd(t-k),k=0..n);
> plot(yd(50,t),t=0..50);
```


0.8
0.6
0.4
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0
–0.2
–0.4
–0.6
10
20
30
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