A1. Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 
 0 & \text{if } 0 \leq t < 2 \\
 2 & \text{if } 4 \leq t. 
\end{cases}$

Solution: Writing $g$ in terms of Heaviside functions, we get

$$g(t) = (t - 2)(u_2(t) - u_4(t)) + 2u_4(t) = (t - 2)(u_2(t)) - (t - 4)u_4(t).$$

Since the Laplace transform of $(t - c)u_c(t)$ is $e^{-cs}$ times the Laplace transform of $t$, we get

$$\mathcal{L}\{g(t)\} = \frac{e^{-2s}}{s^2} - \frac{e^{-4s}}{s^2}.$$

A2. (a) Use integration by parts to prove the following:

**Theorem:** For $n \geq 1$ and $s > 0$, $\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$.

Be sure to point out where you use the hypothesis $s > 0$.

Solution: By definition of the Laplace transform,

$$\mathcal{L}\{t^n\} = \int_{0}^{\infty} e^{-st}t^n \, dt.$$ 

$$= \lim_{A \to \infty} \int_{0}^{A} e^{-st}t^n \, dt.$$ 

We let $u = t^n \, dt$ and $v = e^{-st} \, dt$ and apply integration by parts. We have $du = nt^{n-1} \, dt$ and, since $s \neq 0$, $v = -\frac{e^{-st}}{s}$.

$$\lim_{A \to \infty} \int_{0}^{A} e^{-st}t^n \, dt = \lim_{A \to \infty} \left[ \frac{t^n e^{-st}}{s} \right]_{0}^{A} + \int_{0}^{\infty} nt^{n-1}e^{-st} \, dt$$ 

$$= \lim_{A \to \infty} \left[ 0 + \frac{A^n e^{-sA}}{s} \right] + \frac{n}{s} \int_{0}^{\infty} t^{n-1}e^{-st} \, dt.$$
By the definition of the Laplace transform, the second term on the right is
$$\frac{n}{s}L\{t^{n-1}\}$$. The limit in the first term on the right is
$$\lim_{A \to \infty} \frac{A^n}{e^{sA}}.$$ Since $s > 0$, we have an increasing exponential in the denominator, and a polynomial in the numerator. Since exponentials grow faster than polynomials (see below), we have $\lim_{A \to \infty} \frac{A^n}{e^{sA}} = 0$, and thus the first term on the right is 0. In summary, we have
$$L\{t^n\} = \frac{n}{s}L\{t^{n-1}\}.$$ for $n \geq 1$ and $s > 0$.

Here is a proof that $\lim_{x \to \infty} \frac{x^n}{e^{ax}} = 0$ for $a > 0$ and $n \geq 0$. We proceed by induction on $n$.

**Base case: $n = 0$.** We have $\lim_{x \to \infty} \frac{1}{e^{ax}}$, which is certainly zero, since $e^{ax} \to \infty$ as $x \to \infty$.

**Inductive step.** Let $n \geq 1$, and assume that $\lim_{x \to \infty} \frac{x^{n-1}}{e^{ax}} = 0$. Since $n \geq 1$, we know that $x^n \to \infty$ as $x \to \infty$, and since $a > 0$, we know that $e^{ax} \to \infty$ as well. We may apply l’Hospital’s rule to get
$$\lim_{x \to \infty} \frac{x^n}{e^{ax}} = \lim_{x \to \infty} \frac{n x^{n-1}}{ae^{ax}} = \frac{n}{a} \lim_{x \to \infty} \frac{x^{n-1}}{e^{ax}} = \frac{n}{a} \cdot 0 = 0.$$

(b) Given that $L\{1\} = \frac{1}{s}$ (for $s > 0$), use the formula in part A2a to find $L\{t\}$, $L\{t^2\}$, and $L\{t^3\}$. Based on these results, make a conjecture about the value of $L\{t^n\}$ for any positive integer $n$.

(Optionally, use induction to prove that your conjecture is correct.)
Solution: We have

\[
\mathcal{L}\{t\} = \frac{1}{s} \mathcal{L}\{1\} = \frac{1}{s^2} \quad (s > 0)
\]

\[
\mathcal{L}\{t^2\} = \frac{2}{s} \mathcal{L}\{t\} = \frac{2}{s^3} \quad (s > 0)
\]

\[
\mathcal{L}\{t^3\} = \frac{3}{s} \mathcal{L}\{t^2\} = \frac{6}{s^4} \quad (s > 0).
\]

Each time we increase the exponent on \( t \), we multiply the denominator of the transform by \( s \) and the numerator by the new exponent. We conjecture that

\[
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad (s > 0).
\]

Here's an inductive proof.

We are given that \( \mathcal{L}\{t^0\} = \frac{1}{s} \), so the conjecture holds for \( n = 0 \).

Now suppose that \( n \geq 1 \) and \( \mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n} \) for \( s > 0 \). We want to show that

\[
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{for} \quad s > 0.
\]

By the result above, we have

\[
\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\} = \frac{n}{s} \frac{(n-1)!}{s^n} = \frac{n!}{s^{n+1}},
\]

valid for \( s > 0 \). This completes the proof.
B1. Use Laplace transforms to solve the initial value problem

\[ y'' + 5y' + 6y = g(t), \quad y(0) = y'(0) = 0, \]

where

\[ g(t) = \begin{cases} 
6 & \text{if } 0 \leq t < 1 \\
-6 & \text{if } 1 \leq t < 2 \\
0 & \text{if } 2 \leq t. 
\end{cases} \]

Then use a computer to plot your solution.

Solution: Writing \( g(t) \) in terms of Heaviside functions, we get

\[ g(t) = 6(1 - 2u_1(t) + u_2(t)). \]

Taking the Laplace transform of both sides of the given equation yields

\[ (s^2 + 5s + 6)Y = \frac{6}{s}(1 - 2e^{-t} + e^{-2t}) \]

where \( Y \) is the Laplace transform of our solution. Solving for \( Y \), we get

\[ Y = \frac{6}{s(s^2 + 5s + 6)}(1 - 2e^{-t} + e^{-2t}). \]

It will be convenient to set

\[ H(s) = \frac{6}{s(s^2 + 5s + 6)} \]

and let \( h(t) \) denote the inverse Laplace transform of \( H(s) \).

The denominator of \( H(s) \) factors as \( s(s + 2)(s + 3) \), and using partial fractions, we find that

\[ H(s) = \frac{1}{s} - \frac{3}{s + 2} + \frac{2}{s + 3} \]

so that

\[ h(t) = 1 - 3e^{-2t} + 2e^{-3t}. \tag{1} \]

It follows that our solution \( y(t) \), the inverse Laplace transform of \( Y(s) \) is given by

\[ y(t) = h(t) - 2u_1(t)h(t - 1) + u_2(t)h(t - 2) \]
where \( h(t) \) is given in equation (1).

Here is the solution, plotted by Maple:

\[
\begin{align*}
hb := t &\rightarrow 1 - 3\exp(-2t) + 2\exp(-3t) \\
yb := t &\rightarrow hb(t) - 2\text{Heaviside}(t-1)*hb(t-1) + \\
&\text{Heaviside}(t-2)*hb(t-2) \\
\text{plot}(yb(t), t=0..4);
\end{align*}
\]

B2. (a) Line 19 of the Laplace transform table says that if \( F(s) = \mathcal{L}\{f(t)\} \), then

\[
\mathcal{L}\{tf(t)\} = -F'(s).
\]

Use this fact to show that

\[
\mathcal{L}\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.
\]
Solution: Given \( f(t) = \cos at \) and \( F(s) = \frac{s}{s^2 + a^2} \), we have

\[
\mathcal{L}\{tf(t)\} = -F'(s) = -\frac{(s^2 + a^2) - 2s^2}{(s^2 + a^2)^2} = \frac{s^2 - a^2}{(s^2 + a^2)^2};
\]
as required.

(b) Use the result from part (B2a) and the Laplace transform table to show that

\[
\mathcal{L}\left\{\frac{1}{a} \sin at - t \cos at\right\} = \frac{2a^2}{(s^2 + a^2)^2}.
\]

Solution: We have

\[
\mathcal{L}\left\{\frac{1}{a} \sin at - t \cos at\right\} = \frac{1}{a} \left( \frac{a}{s^2 + a^2} \right) - \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{1}{s^2 + a^2} - \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{2a^2}{(s^2 + a^2)^2},
\]
as required.

(c) Use Laplace transforms to solve the initial value problem

\[
y'' + 4y = g(t), \quad y(0) = y'(0) = 0
\]

where

\[
g(t) = \begin{cases} 
\sin 2t & \text{if } 0 \leq t < 4\pi \\
0 & \text{if } 4\pi \leq t.
\end{cases}
\]

Use a computer to plot your solution.

Solution: Writing \( g(t) \) in terms of Heaviside functions, we get

\[
g(t) = \sin 2t(1 - u_{4\pi}(t)) = \sin 2t - u_{4\pi}(t) \sin 2t = \sin 2t - u_{4\pi}(t) \sin(2(t - 4\pi)),
\]
where the last equality follows because sine is obligingly $2\pi$-periodic (and therefore $8\pi$-periodic).

Taking the Laplace transforms of both sides of our equation, we get

\[(s^2 + 4)Y = (1 - e^{-4\pi s}) \frac{2}{s^2 + 4},\]

where $Y$ is the Laplace transform of the solution. Solving for $Y$, we get

\[Y = (1 - e^{-4\pi s}) H(s)\]  \hspace{1cm} (2)

where

\[H(s) = \frac{2}{(s^2 + 4)^2} = \frac{1}{4} \left( \frac{2 \times 2^2}{(s^2 + 2^2)^2} \right).\]

From the results above, we know that the inverse transform $h(t)$ of $H(s)$ is given by

\[h(t) = \frac{1}{4} \left( \frac{1}{2} \sin 2t - t \cos 2t \right) = \frac{\sin 2t}{8} - \frac{t \cos 2t}{4}.\]

Then from equation (2) above, our solution $y$ is given by

\[y(t) = h(t) - u_{4\pi}(t)h(t - 4\pi).\]

Here is a picture

\[hc := t \rightarrow (1/8)*\sin(2*t) - (1/4)*t*\cos(2*t);\]
\[yc := t \rightarrow hc(t) - \text{Heaviside}(t-4*\Pi)*hc(t-4*\Pi);\]
\[\text{plot}(yc(t), t=0..8*\Pi);\]
C1. Let $g$ be the square wave given (for $t \geq 0$) by

$$g(t) = \sum_{n=0}^{\infty} (-1)^{n} u_{n}(t)$$

(a) Graph $g(t)$, either by hand or with the computer. In Maple, the step function is called `Heaviside()`, and the translation is

$$u_{c}(t) = Heaviside(t - c)$$

Solution: Here is a graph of $g(t)$. *Maple* has included vertical lines at all the points of discontinuity. It would.

```maple
> sq := (n,t) -> sum((-1)^k*Heaviside(t-k),k=0..n);
> plot(sq(10,t),t=0..10,scaling=CONSTRAINED);
```
(b) Let \( G(s) \) be the Laplace transform of \( g(t) \). Find the simplest expression you can for \( G(s) \). Don’t forget to indicate the set of \( s \) for which \( G(s) \) converges.

You may want to use the fact that \( \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \) for \( |r| < 1 \).

Solution: Since the Laplace transform is linear, we have

\[
\mathcal{L} \{ g(t) \} = \sum_{n=0}^{\infty} (-1)^n \mathcal{L} \{ u_n(t) \} 
\]

\[
= \sum_{n=0}^{\infty} (-1)^n e^{-ns} \frac{s}{s} 
\]

\[
= \frac{1}{s} \sum_{n=0}^{\infty} (-e^{-s})^n. 
\]

For \( s > 0 \), we have \( | -e^{-s}| = e^{-s} < 1 \), so the geometric series converges, and we
\[
\mathcal{L}\{g(t)\} = \frac{1}{s} \cdot \frac{1}{1 + e^{-s}}, \quad s > 0.
\]

**C2.** Solve the initial value problem \(y'' + 8y = g(t); \quad y(0) = 0, \quad y'(0) = 0\) where \(g(t)\) is the square wave in problem C1. Plot your solution for \(t\) from 0 to at least 50.

(Note: it will be easier to leave the transform of \(g(t)\) in the form of an infinite sum.)

**Solution:** Taking the transforms of both sides of the differential equation, we get

\[
(s^2 + 8)Y = \frac{1}{s} \sum_{n=0}^{\infty} (-1)^n e^{-ns}
\]

so that

\[
Y = \frac{1}{s(s^2 + 8)} \sum_{n=0}^{\infty} (-1)^n e^{-ns}
\]

Let \(H(s) = \frac{1}{s(s^2 + 8)}\). By partial fractions, we find that

\[
H(s) = \frac{1}{8} \left( \frac{1}{s} - \frac{s}{s^2 + 8} \right).
\]

Let \(h(t)\) be the inverse transform of \(H(s)\). We have

\[
h(t) = \frac{1}{8} - \frac{1}{8} \cos(\sqrt{8}t).
\]

The solution to the IVP is

\[
\sum_{n=0}^{\infty} (-1)^n u_n(t) h(t - n).
\]

Here is a plot of this solution.

```maple
> hd := t -> (1/8)*(1-cos(t*sqrt(8)));
> yd := (n,t) -> sum((-1)^k*Heaviside(t-k)*hd(t-k),k=0..n);
> plot(yd(50,t),t=0..50);
```