Math 333 Problem Set #9 – Solution 6 May 2003 (Optional)

A1. Let $g(t) = \sum_{n=0}^{\infty} (-1)^n \delta(t - \pi n)$.

(a) Solve the IVP $y'' + y = g(t)$; $y(0) = 0$, $y'(0) = 0$.
Solution: Taking Laplace transforms, we get

$$(s^2 + 1)Y(s) = \sum_{n=0}^{\infty} (-1)^n e^{-n\pi s}$$

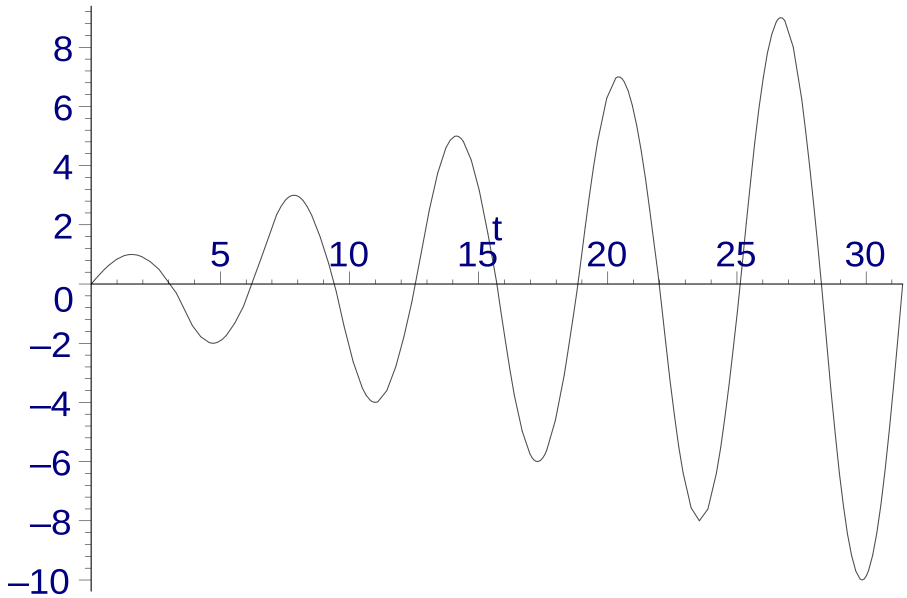
so that

$$Y(s) = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-n\pi s}}{s^2 + 1},$$

and thus

$$y(t) = \sum_{n=0}^{\infty} (-1)^n u_{n\pi} \sin(t - n\pi).$$

(b) Use a computer to plot your solution for at least $0 \leq t \leq 6\pi$.
Solution: Here is the picture



(c) What is being modelled here? Predict the long-term behavior of the system.

Solution: We could think of this as a mass-and spring system in which the mass is given a push every time it passes through its equilibrium position. The push is in the positive direction on even multiples of π and in the negative direction on odd multiples of π, so it's always in phase with the motion of the mass. The amplitude of the response gets larger with every push.

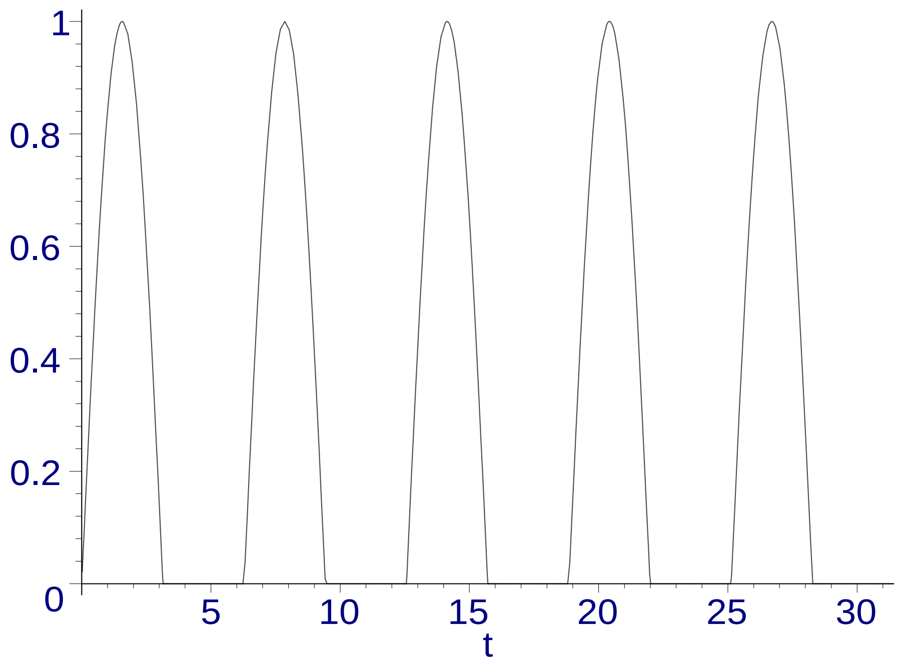
A2. Repeat problem A1 with $g(t) = \sum_{n=0}^{\infty} \delta(t - n\pi)$.

Solution:

(a) The solution this time is

$$y(t) \quad = \quad \sum_{n=0}^{\infty} u_{n\pi} \sin(t - n\pi).$$

(b) Here's the picture



(c) This time the mass is given a push in the same direction every time it passes through its equilibrium position. The push at $t = \pi$ (and each odd multiple of π) is just adequate to bring the mass to a stop. The push at $t = 0$ (and every even multiple of π) starts it moving again.

B1. (a) Solve the initial value problem

$$y'' + 4y' + 13y = -5\delta(t-1) + 3\delta(t-2); \quad y(0) = 0, \quad y'(0) = 1.$$

Solution: Taking Laplace transforms, we get

$$(s^2 - 1)Y(s) + 4sY(s) + 13Y(s) \quad = \quad -5e^{-t} + 3e^{-2t}.$$

so that

$$Y(s) = \frac{1}{s^2+4s+13}(1-5e^{-t}+3e^{-2t}).$$

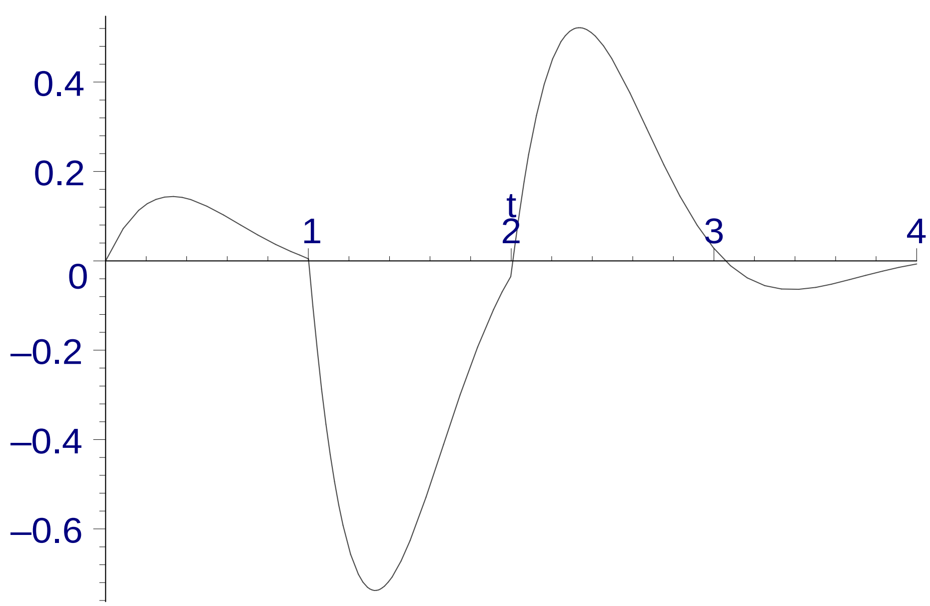
We let $H(s) = \dfrac{1}{s^2+4s+13} = \dfrac{1}{(s+2)^2+9}$. If $h(t) = \mathcal{L}^{-1}\{H(s)\}$, then we have

$$h(t) = \frac{1}{3}e^{-2t}\sin 3t.$$

The solution to the IVP is

$$\begin{aligned} y(t) &= h(t) - 5u_1(t)h(t-1) + 3u_2(t)h(t-2) \\ &= \frac{1}{3}\left(e^{-2t}\sin 3t - 5u_1(t)e^{-2(t-1)}\sin(3(t-1))\right. \\ &\qquad \left. +3u_2(t)e^{-2(t-2)}\sin(3(t-2))\right). \end{aligned}$$

(b) Use the computer to plot your solution.
Solution:



(c) Let y denote the solution to the IVP in B1a. From the picture in part B1b, you might guess that y is not differentiable at $t = 1$ or $t = 2$. However, y is continuous and (twice) differentiable everywhere else, so we can calculate one-sided derivatives at $t = 1$ and $t = 2$.

Find $\lim_{t\to 1^-} y'(t)$, $\lim_{t\to 1^+} y'(t)$, $\lim_{t\to 2^-} y'(t)$, and $\lim_{t\to 2^+} y'(t)$.

Solution: For $t < 1$, $y(t) = \frac{1}{3}e^{-2t}\sin(3t)$, so the left-hand derivative at $t = 1$ is just

$$\frac{d}{dt}\left(\frac{1}{3}e^{-2t}\sin(3t)\right)$$

evaluated at $t = 1$. We get

$$\frac{d}{dt}\left(\frac{1}{3}e^{-2t}\sin(3t)\right) = -\frac{2}{3}e^{-2t}\sin(3t) + e^{-2t}\cos(3t),$$

so

$$\begin{aligned}\lim_{t\to 1^-} y'(t) &= -\frac{2}{3}e^{-2}\sin 3 + e^{-2}\cos 3 \\ &\approx -0.1467.\end{aligned}$$

For $1 < t < 2$, we have

$$y(t) = \frac{1}{3}(e^{-2t}\sin(3t) - 5e^{-2(t-1)}\sin(3(t-1))),$$

so that

$$\begin{aligned}y'(t) = \frac{1}{3}(&-2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t) && (1)\\ &+10e^{-2(t-1)}\sin(3(t-1)) && (2)\\ &-15e^{-2(t-1)}\cos(3(t-1))). && (3)\end{aligned}$$

Subsituting $t = 1$ into this expression gives

$$\begin{aligned}\lim_{t\to 1^+} y'(t) &= \frac{1}{3}(-2e^{-2}\sin 3 + 3e^{-2}\cos 3 - 15) \\ &\approx -5.1467.\end{aligned}$$

To find the left-hand derivative at $t = 2$, we evaluate (3) at $t = 2$. We get

$$\begin{aligned}\lim_{t\to 2^-} y'(t) &= \frac{1}{3}(-2e^{-4}\sin 6 + 3e^{-4}\cos 6 \\ &\quad +10e^{-2}\sin 3 - 15e^{-2}\cos 3) \\ &\approx 0.7546.\end{aligned}$$

Finally, for $t \geq 2$, we have

$$\begin{aligned}y(t) &= \frac{1}{3}(e^{-2t}\sin 3t - 5e^{-2(t-1)}\sin(3(t-1)) + 3e^{-2(t-2)}\sin(3(t-2))) \\ y'(t) &= \frac{1}{3}(-2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t \\ &\quad +10e^{-2(t-1)}\sin(3(t-1)) - 15e^{-2(t-1)}\cos(3(t-1)) \\ &\quad -6e^{-2(t-2)}\sin(3(t-2)) + 9e^{-2(t-2)}\cos(3(t-2)))\end{aligned}$$

To find $\lim_{t\to 2^+} y'(t)$, we evaluate this last expression at $t = 2$. We get

$$\begin{aligned}\lim_{t\to 2^+} y'(t) &= \frac{1}{3}(-2e^{-4}\sin 6 + 3e^{-4}\cos 6 + 10e^{-2}\sin(3) - 15e^{-2}\cos(3) + 9) \\ &\approx 3.7546.\end{aligned}$$

(d) Compute $\lim_{t\to 1^+} y'(t) - \lim_{t\to 1^-} y'(t)$ and $\lim_{t\to 2^+} y'(t) - \lim_{t\to 2^-} y'(t)$.
How are these numbers related to coefficients in the original differential equation?
Solution: We find that

$$\lim_{t\to 1^+} y'(t) - \lim_{t\to 1^-} y'(t) = -5$$

and

$$\lim_{t\to 2^+} y'(t) - \lim_{t\to 2^-} y'(t) = 3.$$

These are exactly the coefficients on the delta functions in the original differential equation.

C1. (B & D, §5.2, problem 3) Consider the differential equation $y'' - xy' - y = 0$. Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ is a solution.

(a) Find the recurrence relation for the coefficients a_n.

Solution: We first rewrite the equation as

$$y'' - (x-1)y' - y' - y = 0.$$

Given $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, we find that

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n \end{aligned}$$

so that

$$(x-1)y' = \sum_{n=0}^{\infty} n a_n (x-1)^n.$$

We also have

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \\ &= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n. \end{aligned}$$

The differential equation reads

$$\sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+2} - n a_n - (n+1)a_{n+1} - a_n)(x-1)^n = 0.$$

The recurrence relation is thus

$$\begin{aligned} a_{n+2} &= \frac{(n+1)(a_n + a_{n+1})}{(n+1)(n+2)} \\ &= \frac{a_n + a_{n+1}}{n+2}. \end{aligned}$$

(b) Find the first four (non-zero) terms in each of two linearly independent solutions.

Solution: Setting $a_0 = 1$ and $a_1 = 0$, we get

$$a_0 = 1;\ a_1 = 0;\ a_2 = \frac{1}{2};\ a_3 = \frac{1}{6};\ a_4 = \frac{1}{6};$$

so one solution is

$$y_0 = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{6} + \cdots$$

Setting $a_0 = 0$ and $a_1 = 1$, we get

$$a_0 = 0;\ a_1 = 1;\ a_2 = \frac{1}{2};\ a_3 = \frac{1}{2};\ a_4 = \frac{1}{4}$$

so a second solution is

$$y_1 = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} + \frac{(x-1)^4}{4} + \cdots$$

C2. (B & D, §5.2, problems 7 and 17) Consider the differential equation $y'' + xy' + 2y = 0$. Suppose $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution.

(a) Find the recurrence relation for the coefficients a_n.

Solution: We have $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$, so that

$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$

and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, which, when we shift the index, becomes

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n.$$

The differential equation implies that

$$\sum_{n=0}^{\infty}((n+1)(n+2)a_{n+2} + na_n + 2a_n)x^n = 0.$$

The recurrence is therefore

$$\begin{aligned} a_{n+2} &= -\frac{(n+2)a_n}{(n+1)(n+2)} \\ &= -\frac{a_n}{n+1}. \end{aligned}$$

(b) Now impose the initial conditions $y(0) = 4$ and $y'(0) = -1$. Write the first five terms of the (power-series) solution to the resulting initial value problem

Solution: We know $a_0 = y(0) = 4$ and $a_1 = y'(0) = -1$, so we can generate some of the coefficients:

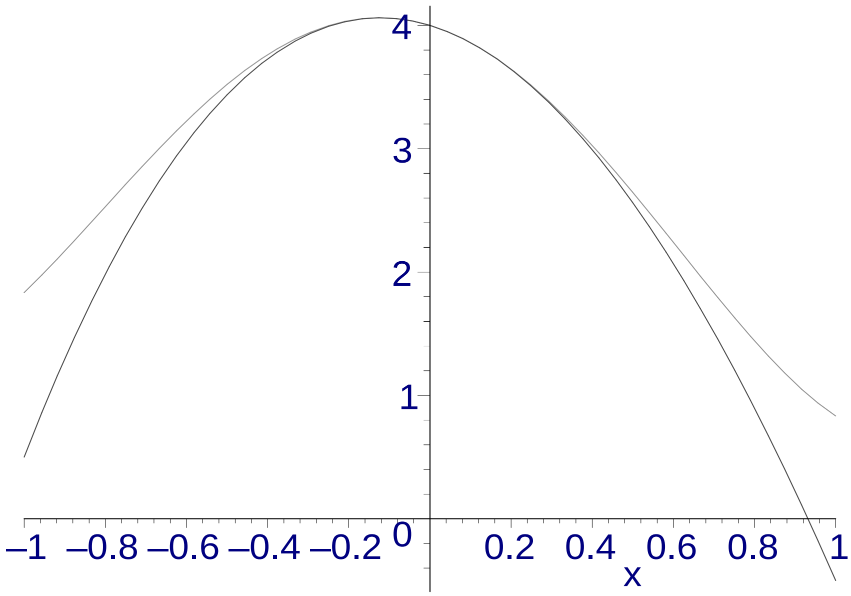
$$a_0 = 4;\ a_2 = -4;\ a_4 = \frac{4}{3};\ a_6 = -\frac{4}{15}$$
$$a_1 = -1;\ a_3 = \frac{1}{2};\ a_5 = -\frac{1}{8};\ a_7 = \frac{1}{48}.$$

The power series solution begins

$$y = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 + \cdots$$

(c) Plot the four-term and five-term approximations to the solution on the same axes. Estimate the interval on which the four-term approximation is reasonably accurate.

Solution: Here is the picture



It's not clear what is meant by "reasonably accurate," but from the picture, we'd guess that the four-term approximation has an error of less than 0.1 for x between about -0.4 and 0.4.