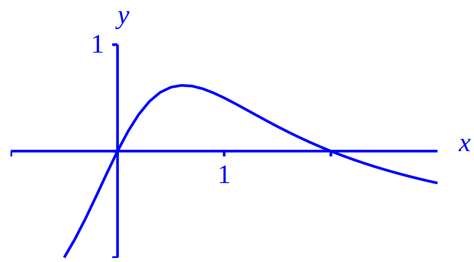
Math 333 Solution
Quiz 3 20 February 2003

1. Consider the differential equation $\dfrac{dy}{dt} = \dfrac{2y - y^2}{1 + y^2}$. Find all equilibrium solutions, and classify each as asymptotically stable or unstable.

   Solution: The differential equation has the form $dy/dt = f(y)$, where

   $$f(y) = \frac{y(2-y)}{1+y^2}.$$

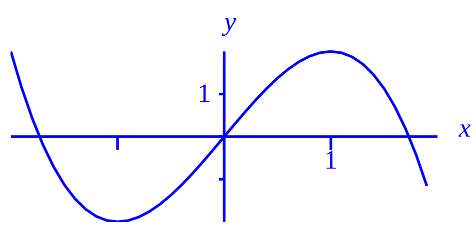
   Clearly, $f(y) = 0$ only when $y = 0$ or $y = 2$, so these are our two equilibrium solutions. To determine their stability, we need only observe that $f(-1)$ is negative, $f(1)$ is positive, and $f(3)$ is negative. Since f changes sign only at $y = 0$ and $y = 2$, it must be that f is positive on the interval $(0, 2)$ and negative elsewhere.



   From this, we conclude that the solution $y = 0$ is unstable and the solution $y = 2$ is asymptotically stable.

2. Suppose $y(t)$ is a solution to the initial value problem $y' = 3y - y^3; \quad y(0) = \dfrac{1}{2}$. Without solving the IVP, find $\lim_{t\to\infty} y(t)$.

   Solution: Let $f(y) = 3y - y^3 = y(3 - y^2)$. Then f is a cubic with roots at $y = 0$ and $y = \pm\sqrt{3}$.



   A solution that begins anywhere in the interval between 0 and $\sqrt{3}$ will increase toward $\sqrt{3}$, so we conclude that

   $$\lim_{t\to\infty} y(t) = \sqrt{3}.$$