Problem: Use Laplace transforms to solve the initial-value problem

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \quad y(0) = 0, \quad y'(0) = \frac{1}{2}.$$

Plot the solution in Maple.

Solution: Taking the Laplace transform of each side, we get

$$\begin{aligned}
(s^2+3s+2)Y - \frac{1}{2} &= e^{-5s} + \frac{e^{-10s}}{s} \\
Y &= \frac{1}{s^2+3s+2}\left(\frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}\right) \\
&= \frac{(1/2)}{s^2+3s+2} + \frac{e^{-5s}}{s^2+3s+2} + \frac{e^{-10s}}{s(s^2+3s+2)}
\end{aligned}$$

where $Y(s)$ is the Laplace transform of the solution, $y(t)$.

Let $H_1(s) = \dfrac{1}{s^2+3s+2}$, and let $h_1(t) = \mathcal{L}^{-1}\{H_1(s)\}$. To compute $h_1(t)$, we use partial fractions to write

$$H_1(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

From this we get immediately that $h_1(t) = e^{-t} - e^{-2t}$.

Let $H_2(s) = \dfrac{1}{s(s^2+3s+2)}$ and let $h_2(t) = \mathcal{L}^{-1}\{H_2(s)\}$. To compute $h_2(t)$, we use partial fractions to write

$$H_2(s) = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

From this we get $h_2(t) = \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t}$.

Now using the properties of exponentials and the Laplace transform, we have

$$\begin{aligned}
\mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{2}\mathcal{L}^{-1}\{H_1(s)\} + \mathcal{L}^{-1}\{e^{-5s}H_1(s)\} + \mathcal{L}^{-1}\{e^{-10s}H_2(s)\} \\
&= \frac{1}{2}h_1(t) + u_5(t)h_1(t-5) + u_{10}(t)h_2(t-10)
\end{aligned}$$

where $h_1(t) = e^{-t} - e^{-2t}$ and $h_2(t) = \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t}$.

To plot this in Maple, we define two functions `h1` and `h2`, and multiply them as needed by the step function, which is called `Heaviside` in Maple. Here's the Maple worksheet:

```
> h1 := t -> exp(-t) - exp(-2*t):
> h2 := t -> 1/2 - exp(-t) + (1/2)*exp(-2*t):
> y := t -> (1/2)*h1(t) +
> Heaviside(t-5)*h1(t-5) +
> Heaviside(t-10)*h2(t-10):
> plot(y(t),t=0..15);
```