Math 202(01) Solution
Hour Exam I 18 – 19 February 2004

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1. Given that $G(x) = \int_1^{\sin(3x)} \sqrt{1-t^2}\,dt$, find $G'(x)$.

   Solution: Let $H(u) = \int_1^{u} \sqrt{1-t^2}\,dt$. Then by the Fundamental Theorem of Calculus (Part I), we know that
   $$H'(u) = \sqrt{1-u^2}$$
   Since $G(x) = H(\sin(3x))$, we can use the chain rule to get
   $$\begin{aligned} G'(x) &= \frac{d}{dx}H(\sin(3x)) \\ &= H'(\sin(3x)) \times \cos(3x) \times 3 \\ &= \sqrt{1-\sin^2(3x)} \times 3\cos(3x) \\ &= 3\cos(3x)\sqrt{\cos^2(3x)} \\ &= 3\cos(3x)|\cos(3x)| \end{aligned}$$

2. Find $\int t^{-1}(2t+1)^2\,dt$.

   Solution: We have
   $$\begin{aligned} \int t^{-1}(2t+1)^2\,dt &= \int t^{-1}(4t^2+4t+1)\,dt \\ &= \int 4t+4+\frac{1}{t}\,dt \\ &= 2t^2+4t+\ln|t|+C \end{aligned}$$

3. Find $\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$.

   Solution: Let $u = 4-x^2$. Then $du = -2x\,dx$, so that $x\,dx = -\frac{1}{2}\,du$.

   When $x = 0$, we have $u = 4$, and when $x = 1$, we have $u = 3$. The integral becomes
   $$\begin{aligned} -\frac{1}{2}\int_4^3 \frac{1}{\sqrt{u}}\,du &= -\frac{1}{2}\left[2\sqrt{u}\right]_4^3 \\ &= -\frac{1}{2}\left(2\sqrt{3}-2\sqrt{4}\right) \\ &= 2-\sqrt{3} \end{aligned}$$

4. Find $\displaystyle\int \frac{x^5}{x^3+2}\,dx$.

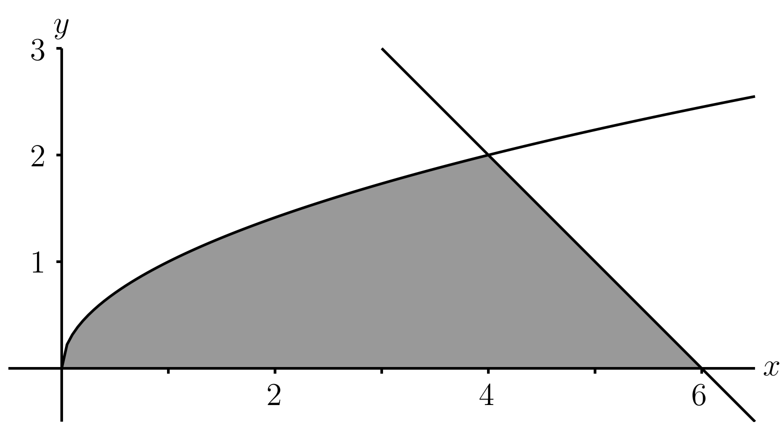
   Solution: Let $u = x^3 + 2$. Then $du = 3x^2\,dx$, so that $\frac{1}{3}du = x^2\,dx$. We also will want to know that $x^3 = u - 2$.

   We can rewrite the integral as

$$\begin{aligned}
\int \frac{x^3}{x^3+2}\,x^2\,dx &= \frac{1}{3}\int \frac{u-2}{u}\,du \\
&= \frac{1}{3}\int 1 - \frac{2}{u}\,du \\
&= \frac{1}{3}\left[u - 2\ln|u|\right] + C \\
&= \frac{x^3+2}{3} - \frac{2}{3}\ln|x^3+2| + C
\end{aligned}$$

5. Find the area of the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x + y = 6$.

   Solution: Here is a sketch showing the curves and the region.



The curve $y = \sqrt{x}$ and the line $x + y = 6$ intersect when $6 - x = \sqrt{x}$, that is, when $x = (6-x)^2$. Multiplying this out and moving all the terms to one side, we get

$$\begin{aligned} 0 &= x^2 - 13x + 36 \\ &= (x-4)(x-9) \end{aligned}$$

We can discard the solution $x = 9$, because that would give us the point $(9, -3)$, and clearly $(9, -3)$ does not lie on the curve $y = \sqrt{x}$. The intersection point in the drawing is $(4, 2)$.

The area is most easily found by integrating with respect to y. A horizontal rectangle at height y in this region has its right edge along the line $x = 6 - y$ and its left edge along the curve $x = y^2$. The integral for the area of the region is thus

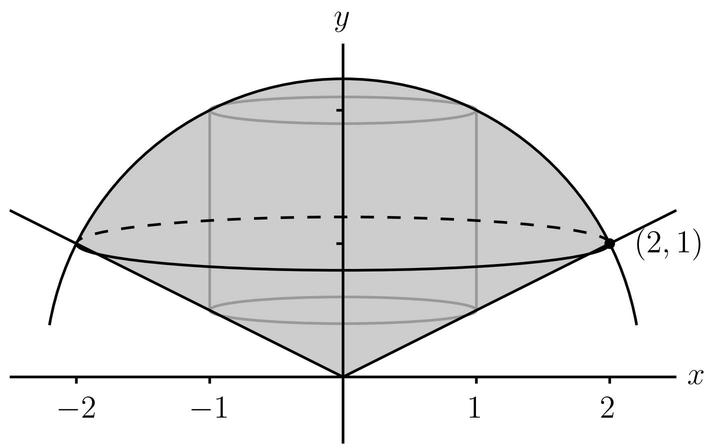
$$\begin{aligned} \int_0^2 (6-y) - y^2\, dy &= 6y - \frac{y^2}{2} - \frac{y^3}{3}\Big]_0^2 \\ &= 12 - 2 - \frac{8}{3} - (0) \\ &= \frac{22}{3} \end{aligned}$$

6. Let R be the region in the first quadrant inside the circle $x^2 + y^2 = 5$ and above the line $2y = x$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is rotated about the y-axis.

   Solution: To find the intersection points of the line and the circle, we can substitute $2y$ for x in the circle equation. We get

   $$(2y)^2 + y^2 = 5$$

   from which we get $5y^2 = 5$, and so the intersection points occur when $y = \pm 1$. The intersection point in the first quadrant is $(2, 1)$.



The integral is simplest if we use the shells method, with shells going from $x = 0$ to $x = 2$. The shell at position x has radius x. Its height is determined by the circle with equation $y = \sqrt{5 - x^2}$ and the line with equation $y = \dfrac{x}{2}$. We have

$$2\pi(\text{radius})(\text{height}) \quad = \quad 2\pi x \left(\sqrt{5-x^2} - \left(\frac{x}{2}\right)\right)$$

so that the volume is given by

$$V \quad = \quad \int_0^2 2\pi x \left(\sqrt{5-x^2} - \left(\frac{x}{2}\right)\right)\, dx$$

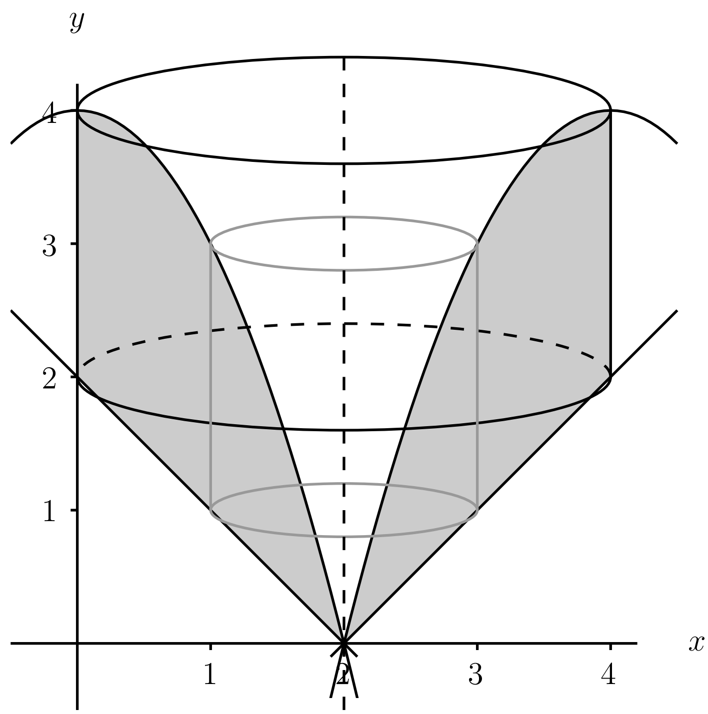
We could also find this volume using the disk method. For y between 0 and 1, the disk at height y has radius $2y$, so its area is $\pi(2y)^2$.

For y between 1 and $\sqrt{5}$ (which is the y-value at the top of the solid), the disk at height y has radius $\sqrt{5 - y^2}$, so its area is $\pi(5 - y^2)$. The volume is given by

$$V = \int_0^1 \pi(2y)^2\, dy + \int_1^{\sqrt{5}} \pi(5 - y^2)\, dy$$

7. Let R be the region to the right of the y-axis, below the curve $y = 4 - x^2$, and above the line $x + y = 2$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is revolved about the line $x = 2$.

   Solution: The line and the parabola intersect only once on the right side of the y-axis, and the intersection point is clearly $(2, 0)$. Here is a picture of the region R and the solid it generates.



The integral will be easiest to set up using shells. With the region R lying to the left of the axis of rotation, we will have shells going from the outermost (at $x = 0$) to the innermost (at $x = 2$). The radius of the shell at position x is the horizontal distance between x and the vertical line at 2, so the radius is $2 - x$. The top of the shell lies on the curve $y = 4 - x^2$, and the bottom lies on the line $y = 2 - x$, so the height of the shell at position x is

$$(4 - x^2) - (2 - x)$$

The volume of the shell at position x is $2\pi(2-x)((4-x^2)-(2-x))\,\Delta x$, so the volume of the entire solid is given by

$$V = \int_0^2 2\pi(2 - x)((4 - x^2) - (2 - x))\,dx$$

We could also set up this volume using the washer method. We need the bounding curves solved for x. The line is $x = 2 - y$, and the relevant part of the parabola is $x = \sqrt{4 - y}$. For y between 0 and 2, the washer at height y has

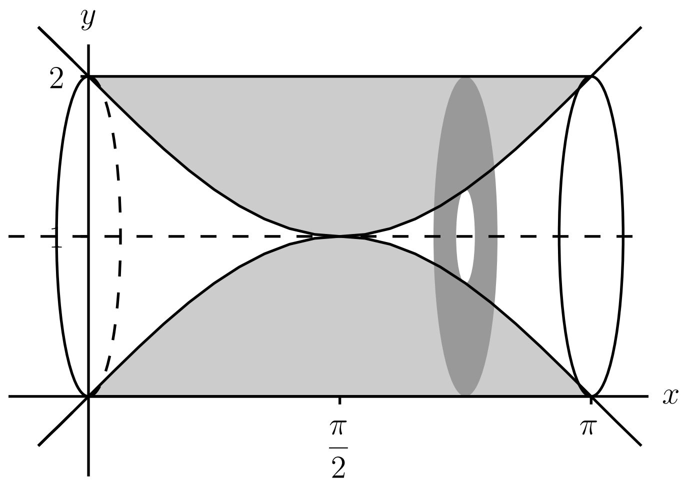
$$\text{outer radius} = 2 - (2 - y) \quad \text{and} \quad \text{inner radius} = 2 - \sqrt{4 - y}$$

For y between 2 and 4, the outer radius is 2 and the inner radius is as above. The volume is given by

$$V = \int_0^2 \pi((2-(2-y))^2 - (2-\sqrt{4-y})^2)\,dy + \int_2^4 \pi(2^2 - (2-\sqrt{4-y})^2)\,dy$$

8. Let R be the region under the curve $y = \sin(x)$ and above the x-axis, between $x = 0$ and $x = \pi$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is revolved about the line $y = 1$.

Solution: Here is a picture of the region and the solid it generates.



The integral is simplest if we use the washer method. The leftmost washer is at $x = 0$ and the rightmost is at $x = \pi$. The washer at position x has center at $y = 1$ and outer edge at $y = 0$, so its outer radius is 1. The inner edge of the washer at position x is determined by $y = \sin(x)$, so its inner radius is $1 - \sin(x)$. The volume of the washer at position x is $\pi(1^2 - (1-\sin(x))^2)$, so the volume of the whole solid is given by

$$V = \int_0^\pi \pi(1 - (1-\sin(x))^2)\,dx$$

Setting this up using shells is a challenge. The shell at position y has radius $1 - y$ (where we take y to be between 0 and 1). The height of the shell is determined by the x-coordinates of the two points where $\sin x = y$. The leftmost of these points has x-coordinate given by

$$x = \sin^{-1} y$$

The rightmost also satisfies $\sin x = y$, but it lies between $x = \frac{\pi}{2}$ and $x = \pi$, so it can't be just $\sin^{-1} y$, because the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We use the fact that the sine curve is symmetric about the line $x = \pi$ to conclude that the x-coordinate at the rightmost point on the shell at position y is given by

$$x = \pi - \sin^{-1} y$$

The volume of the solid is thus

$$\int_0^1 2\pi(1 - y)((\pi - \sin^{-1} y) - \sin^{-1} y)\, dy$$