Math 202(01) — Solution

Hour Exam II — 24 – 25 March 2004

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1. A force of 10 lb stretches a certain spring 6 in beyond its natural length. How much work is done stretching the same spring from its natural length to 18 in beyond its natural length?

   Solution: The given information implies that the spring constant is 20 lb/ft. We stretch the spring from 0 to 3/2 feet beyond its natural length, so the work done is

$$\begin{aligned} W &= \int_0^{\frac{3}{2}} 20x\,dx \\ &= \left[10x^2\right]_0^{\frac{3}{2}} \\ &= 10 \times \frac{9}{4} \\ &= \frac{45}{2} \text{ lb-ft} \end{aligned}$$

2. A uniform cable, 20 meters long and with a total mass of 10 kg, hangs from the edge of the roof of a high building. How much work is required to lift the entire cable to roof level? Use 9.8 m/s$^2$ for gravitational acceleration.

   Solution: The cable's mass density is 0.5kg/m, so when $x$ meters have been pulled up, the remaining cable has a mass of $0.5(20-x)$ kg, and thus a weight of

$$9.8 \times 0.5(20-x) \text{ N} \quad = \quad 4.9(20-x) \text{ N}$$

   The work required to pull the entire cable onto the roof is

$$\begin{aligned} \int_0^{20} 4.9(20-x)\,dx &= \left[98x - \frac{4.9x^2}{2}\right]_0^{20} \\ &= 980 \text{ N-m} \end{aligned}$$

3. Evaluate $\int_1^4 x^3 \ln x\,dx$.

   Solution: We use integration by parts, with

$$\begin{aligned} u &= \ln x & \qquad v &= \frac{x^4}{4} \\ du &= \frac{1}{x}\,dx & \qquad dv &= x^3\,dx \end{aligned}$$

We get

$$\begin{aligned}
\int_1^4 x^3 \ln x \, dx &= \left[\frac{x^4 \ln x}{4}\right]_1^4 - \int_1^4 \frac{x^3}{4} \, dx \\
&= 64 \ln 4 - \left[\frac{x^4}{16}\right]_1^4 \\
&= 64 \ln 4 - \left(16 - \frac{1}{16}\right) \\
&= 64 \ln 4 - \frac{255}{16}
\end{aligned}$$

4. Compute $\int \tan^5 x \sec^4 x \, dx$

   Solution: We write

$$\begin{aligned}
\int \tan^5 x \sec^4 x \, dx &= \int \tan^5 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx
\end{aligned}$$

and let $u = \tan x$ so that $du = \sec^2 x \, dx$. We get

$$\begin{aligned}
\int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx &= \int u^5 (1 + u^2) \, du \\
&= \int u^5 + u^7 \, du \\
&= \frac{u^6}{6} + \frac{u^8}{8} + C \\
&= \frac{\tan^6 x}{6} + \frac{\tan^8}{8} + C
\end{aligned}$$

5. Compute $\int \frac{1}{(9 - x^2)^{\frac{3}{2}}} \, dx$

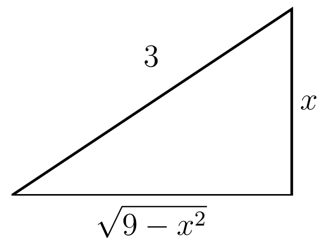
   Solution: We use the substitution $x = 3 \sin \theta$, so that $dx = 3 \cos \theta \, d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{1}{(9 - 9 \sin^2 \theta)^{\frac{3}{2}}} 3 \cos \theta \, d\theta &= \int \frac{3 \cos \theta}{(9 \cos^2 \theta)^{\frac{3}{2}}} \, d\theta \\
&= \int \frac{3 \cos \theta}{27 \cos^3 \theta} \, d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{9}\int \sec^2\theta\, d\theta \\
&= \frac{1}{9}\tan\theta + C
\end{aligned}
$$

From $\sin\theta = \dfrac{x}{3}$, we get $\tan\theta = \dfrac{x}{\sqrt{9-x^2}}$, so the answer is

$$\frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C$$




6. Compute $\displaystyle\int \frac{x^3+1}{x+5}\,dx$

We carry out the indicated long division. We get

$$
\begin{array}{r|llllllll}
 & x^2 & - & 5x & + & 25 & & \\
\hline
x+5 & x^3 & & & & & + & 1 \\
 & x^3 & + & 5x^2 & & & & \\
\cline{2-4}
 & & - & 5x^2 & & & & \\
 & & - & 5x^2 & - & 25x & & \\
\cline{4-6}
 & & & & & 25x & & \\
 & & & & & 25x & + & 125 \\
\cline{6-8}
 & & & & & & - & 124
\end{array}
$$

The integrand is equal to $x^2 - 5x + 25 - \dfrac{124}{x+5}$, so the integral is

$$\frac{x^3}{3} - \frac{5x^2}{2} + 25x - 124\ln|x+5| + C$$

7. Compute $\displaystyle\int \frac{3x^2+x+2}{x(x^2+1)}\,dx$

Solution: We use partial fractions. We have

$$\frac{3x^2+x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Clearing denominators, we get

$$\begin{aligned} 3x^2 + x + 2 &= A(x^2+1) + (Bx+C)x \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

This implies that $A = 2$, $C = 1$, and $A + B = 3$, so $B$ must be 1, as well. We have

$$\begin{aligned} \int \frac{2}{x} + \frac{x+1}{x^2+1}\,dx &= \int \frac{2}{x} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= 2\ln|x| + \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + C \end{aligned}$$

8. Use the midpoint rule with $n = 20$ subintervals to estimate $\int_0^1 \sqrt{1-x^2}\,dx$. Give your answer to eight decimal places.

Solution: The subintervals are

$$\left[0, \frac{1}{20}\right], \left[\frac{1}{20}, \frac{2}{20}\right], \ldots, \left[\frac{19}{20}, 1\right]$$

so their midpoints are the numbers

$$\frac{1}{40}, \frac{3}{40}, \ldots, \frac{39}{40}$$

Let $f(x) = \sqrt{1-x^2}$. The midpoint estimate for this integral is

$$\frac{1}{20}\left[f\left(\frac{1}{40}\right) + f\left(\frac{3}{40}\right) + \cdots + f\left(\frac{39}{40}\right)\right]$$

Keying this into the calculator, we get

$$M_{20} \approx 0.78635765$$