Math 202(01) Solution
Hour Exam III 22 – 23 April 2004

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1. Use Simpson's rule with $n = 8$ to estimate the value of $\int_0^2 e^{-x^2}\,dx$. Round your answer to eight decimal places.

   Solution: The subintervals are

$$\left[\frac{0}{4},\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right],\ldots,\left[\frac{7}{4},\frac{8}{4}\right]$$

   Writing $f(x) = e^{-x^2}$, we have the approximation

$$\frac{1}{3\times 4}\left[f(1)+4f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+4f\left(\frac{3}{4}\right)+\cdots+2f\left(\frac{6}{4}\right)+4f\left(\frac{7}{4}\right)+f(2)\right]$$

   The calculator says this is about 0.88206551.

2. Use an integral to find the surface area of the band generated when the part of the line $y = \frac{1}{2}x$ with $2 \le x \le 4$ is revolved about the x-axis.

   Solution: We have $y' = \frac{1}{2}$, so that the surface area of the band is given by

$$\begin{aligned}
\int_2^4 2\pi y\sqrt{1+(y')^2}\,dx &= \int_2^4 2\pi\frac{x}{2}\sqrt{1+\frac{1}{4}}\,dx \\
&= \pi\sqrt{\frac{5}{4}}\int_2^4 x\,dx \\
&= \pi\sqrt{\frac{5}{4}}\left[\frac{x^2}{2}\right]_2^4 \\
&= \pi\sqrt{\frac{5}{4}}\,[8-2] \\
&= 6\pi\sqrt{\frac{5}{4}} \\
&= 3\pi\sqrt{5}
\end{aligned}$$

3. Evaluate $\int_0^4 \frac{x}{x-2}\,dx$

Solution: Since the denominator is 0 at $x = 2$, we have an improper integral. We write

$$\begin{aligned}\int_0^4 \frac{x}{x-2}\,dx &= \int_0^2 \frac{x}{x-2}\,dx + \int_2^4 \frac{x}{x-2}\,dx \\ &= \lim_{t\to 2^-}\int_0^t \frac{x}{x-2}\,dx + \lim_{t\to 2^+}\int_t^4 \frac{x}{x-2}\,dx\end{aligned}$$

We first evaluate the indefinite integral $\int \frac{x}{x-2}\,dx$. We get

$$\begin{aligned}\int \frac{x}{x-2}\,dx &= \int \frac{x-2+2}{x-2}\,dx \\ &= \int 1 + \frac{2}{x-2}\,dx \\ &= x + 2\ln|x-2| + C\end{aligned}$$

The first of the limits above is then

$$\lim_{t\to 2^-}\left[x + 2\ln|x-2|\right]_0^2 = \lim_{t\to 2^-}(t + \ln|t-2| - \ln 2)$$

The middle term $\ln|t-2|$ goes to $-\infty$ as $t \to 2^-$, so this part of the integral diverges. Therefore the entire integral is divergent.

4. Rewrite the integral $\int_1^\infty \frac{\ln x}{x^2}\,dx$ as a limit, and then evaluate it.

Solution: We have

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx$$

To evaluate the indefinite integral $\int \frac{\ln x}{x^2}\,dx$, we try integration by parts with

$$\begin{aligned} u &= \ln x & v &= -\frac{1}{x} \\ du &= \frac{1}{x}\,dx & dv &= \frac{1}{x^2}\,dx\end{aligned}$$

We get

$$\begin{aligned}
\int \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx \\
&= -\frac{\ln x}{x} - \frac{1}{x} + C \\
&= -\frac{1+\ln x}{x} + C
\end{aligned}$$

Evaluating the limit, we get

$$\begin{aligned}
\lim_{t\to\infty}\left[-\frac{1+\ln x}{x}\right]_1^t &= \lim_{t\to\infty}\left(-\frac{1+\ln t}{t} + \frac{1+\ln 1}{1}\right) \\
&= 1 - \lim_{t\to\infty}\frac{1+\ln t}{t} \\
&= 1 - \lim_{t\to\infty}\frac{0+\frac{1}{t}}{1} \quad \text{by l'Hospital} \\
&= 1
\end{aligned}$$

5. Suppose that the waiting times in the line at the post office are exponentially distributed with a mean waiting time of 2 minutes. What fraction of people who go to the post office have to wait in line for more than 2 minutes?

   Give your answer in exact form and as a decimal approximation.

   Solution: We know that the probability density function for these waiting times is $\frac{1}{2}e^{-\frac{t}{2}}$ (for $t > 0$), so the probability we want is just

$$\begin{aligned}
\int_2^\infty \frac{1}{2}e^{-\frac{t}{2}}\,dt &= 1 - \int_0^2 \frac{1}{2}e^{-\frac{t}{2}}\,dt \\
&= 1 - \left[-e^{\frac{t}{2}}\right]_0^2 \\
&= 1 - (1 - e^{-1}) \\
&= e^{-1} \\
&\approx 0.3679
\end{aligned}$$

6. Find the exact value of the sum of the geometric series that begins

$$3 - \frac{18}{7} + \frac{108}{49} - \frac{648}{343} + \cdots$$

Solution: Since we're told that this is a geometric series, we can find everything we need to know from just the first two terms. We have $a = 3$ (the first term) and $r = \dfrac{-18/7}{3} = -\dfrac{6}{7}$. Thus the sum of the series is given by

$$\frac{3}{1+\frac{6}{7}} = \frac{21}{13}$$

7. Determine whether each of the following series is convergent or divergent. Give reasons for your conclusions.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n^2}$

Solution: We can apply the Basic Comparison Theorem. We know that

$$\begin{aligned} \frac{n+1}{2n^2} &> \frac{n}{2n^2} \\ &= \frac{1}{2n} \end{aligned}$$

and that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent.

Thus the given series diverges by Basic Comparison with the harmonic series.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+2)}$

Solution: We apply the Alternating Series Test. The function $\ln(n+2)$ is increasing, and $\displaystyle\lim_{n\to\infty} \ln(n+2) = \infty$, so that $\dfrac{1}{\ln(n+2)}$ is decreasing and $\displaystyle\lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$.

Thus the given series converges by the Alternating Series Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2}}{n}$

Solution: We note that

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{n^2+2}}{n} &= \lim_{n\to\infty} \sqrt{\frac{n^2+2}{n^2}} \\ &= \sqrt{\lim_{n\to\infty} \frac{n^2+2}{n^2}} \\ &= \sqrt{1} \\ &\neq 0 \end{aligned}$$

so this series diverges by the Divergence Test.

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

Solution: We apply the Integral Test. We have

$$\int_3^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t \to \infty} \int_3^t \frac{1}{x \ln x}\, dx$$

We use the substitution $u = \ln x$, $du = \dfrac{1}{x}\, dx$ to get

$$\begin{aligned} \int \frac{1}{x \ln x}\, dx &= \int \frac{1}{u}\, du \\ &= \ln u + C \\ &= \ln \ln x + C \end{aligned}$$

The improper integral then becomes

$$\lim_{t \to \infty} \ln \ln t - \ln \ln 3$$

which diverges.

The original series is divergent, by the Integral Test.