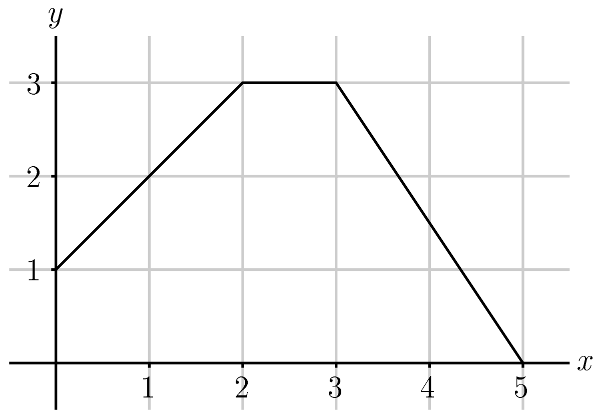
1. Let f be the function whose graph is shown at right. Let

$$F(x) = \int_0^{\frac{1}{x}} f(t)\,dt$$

Find $F'\left(\frac{1}{2}\right)$.



Solution: Write $G(x) = \int_0^x f(t)\,dt$. Then $F(x) = G\left(\frac{1}{x}\right)$, so

$$F'(x) = G'\left(\frac{1}{x}\right) \times -\frac{1}{x^2}.$$

Thus

$$\begin{aligned} F'\left(\frac{1}{2}\right) &= G'(2) \times -\frac{1}{(1/2)^2} \\ &= -4f(2) \\ &= -12 \end{aligned}$$

Alternate Solution: For $t \leq 2$, it appears that $f(t) = t + 1$, so that

$$\begin{aligned} F(x) &= \int_0^{\frac{1}{x}} t + 1\,dt \\ &= \left.\frac{t^2}{2} + t\right]_0^{\frac{1}{x}} \\ &= \frac{1}{2x^2} + \frac{1}{x} \end{aligned}$$

for values of $x \geq \frac{1}{2}$. Thus for $x > \frac{1}{2}$, we have

$$F'(x) = -\frac{1}{x^3} - \frac{1}{x^2}$$

and the right-hand derivative at $x = \dfrac{1}{2}$ is

$$\begin{aligned} F'_+\left(\frac{1}{2}\right) &= -8 - 4 \\ &= -12 \end{aligned}$$

For $2 \le t \le 3$, it appears that $f(t) = 3$, so that

$$\begin{aligned} F(x) &= 4 + \int_2^{\frac{1}{x}} 3\,dt \\ &= 4 + \frac{3}{x} \end{aligned}$$

for $\dfrac{1}{3} \le x \le \dfrac{1}{2}$. Thus for $\dfrac{1}{3} < x < \dfrac{1}{2}$, we have

$$F'(x) = -\frac{3}{x^2}$$

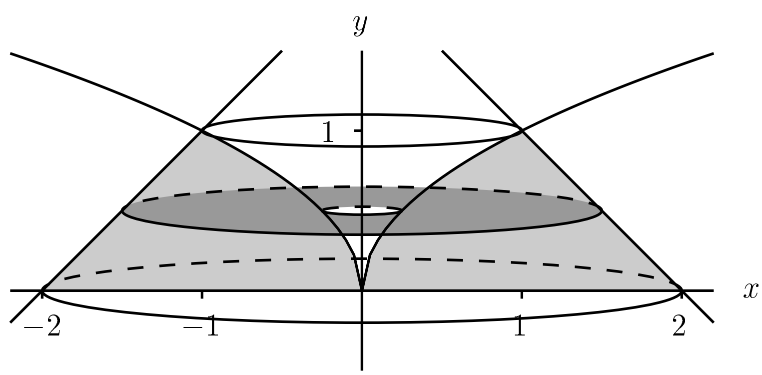
and the left-hand derivative at $x = \dfrac{1}{2}$ is

$$\begin{aligned} F'_-\left(\frac{1}{2}\right) &= -3 \times 4 \\ &= -12 \end{aligned}$$

Since the left- and right-hand derivatives of F at $x = \dfrac{1}{2}$ are equal, F is differentiable at $x = \dfrac{1}{2}$, and the value of the derivative is -12.

2. Let R be the region bounded by the curve $y = \sqrt{x}$, the line $y = 2 - x$, and the x-axis. Set up, but do not evaluate, an integral for the volume of the solid generated when R is revolved about the y-axis.

Solution: Here is a sketch of the solid, along with a typical washer.



The line and the curve intersect when $\sqrt{x} = 2 - x$, that is, when

$$\begin{aligned} x &= 4 - 4x + x^2 \\ x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0 \end{aligned}$$

Clearly, the solution we are looking for is $x = 1$; the point of intersection is $(1, 1)$.

The volume is most easily set up using washers. The washer at height y has center at $x = 0$. Its outer edge is on the line $y = 2 - x$, so that we have $x = 2 - y$ at the outer edge, and the outer radius is $(2 - y) - 0$, or just $2 - y$. The inner edge of the washer is on the curve $y = \sqrt{x}$, so that we have $x = y^2$ at the inner edge, and the inner radius of the washer is $y^2 - 0$, or just y^2.

The volume is given by

$$\int_0^1 \pi((2-y)^2 - (y^2)^2)\, dy$$

We could have set this up using shells centered on the y-axis. The shells with radius x for $0 \le x < 1$ have height given by $y = \sqrt{x}$. The shells with radius x for $1 \le x \le 2$ have height given by $y = 2 - x$. The volume is given by

$$\int_0^1 2\pi x\sqrt{x}\, dx + \int_1^2 2\pi x(2-x)\, dx$$

3. Evaluate the following.

   (a) $\displaystyle\int \frac{dx}{(x^2-4)^{\frac{3}{2}}}$

   Solution: We try the trig substitution

$$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\, d\theta$$

   We get

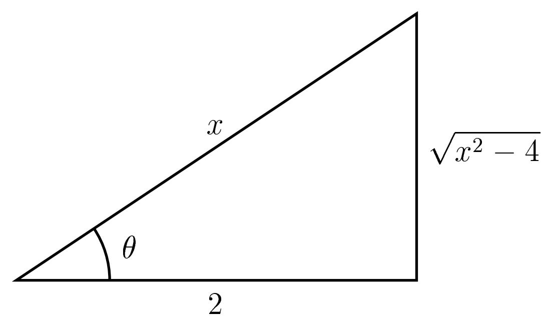
$$\begin{aligned} \int \frac{dx}{(x^2-4)^{\frac{3}{2}}} &= \int \frac{2\sec\theta\tan\theta}{(4\sec^2\theta - 4)^{\frac{3}{2}}}\, d\theta \\ &= \int \frac{2\sec\theta\tan\theta}{(4\tan^2\theta)^{\frac{3}{2}}}\, d\theta \end{aligned}$$

$$\begin{aligned}
&= \int \frac{2\sec\theta\tan\theta}{8\tan^3\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta
\end{aligned}$$

Now we let $u = \sin\theta$, so that $du = \cos\theta\,d\theta$. We get

$$\begin{aligned}
\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta &= \frac{1}{4}\int u^{-2}\,du \\
&= -\frac{1}{4}u^{-1} + C \\
&= -\frac{1}{4\sin\theta} + C
\end{aligned}$$

Using the fact that $x = 2\sec\theta$, we draw the figure at right. From this, we conclude that $\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$.



Thus we have

$$\int \frac{dx}{(x^2-4)^{\frac{3}{2}}} = -\frac{x}{4\sqrt{x^2-4}} + C.$$

(b) $\displaystyle\int_0^3 \frac{dx}{x^2+x-2}$

Solution: To begin, we factor the denominator. We have

$$\int_0^3 \frac{dx}{x^2+x-2} = \int_0^3 \frac{dx}{(x-1)(x+2)}$$

We apply the method of partial fractions. We have

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{1}{(x-1)(x+2)}$$

for some unknown constants A and B. Clearing denominators, we get the equation

$$Ax + 2A + Bx - B = 1$$

which implies that $A+B=0$ and $2A-B=1$. The solution to this system is

$$A=\frac{1}{3} \quad \text{and} \quad B=-\frac{1}{3}$$

We now have

$$\int_0^3 \frac{dx}{x^2+x-2} = \frac{1}{3}\int_0^3 \frac{1}{x-1}\,dx - \frac{1}{3}\int_0^3 \frac{1}{x+2}\,dx$$

The first of these is an improper integral, since the denominator is 0 at $x=1$, and $x=1$ falls within the interval of integration. We write

$$\begin{aligned}\int_0^3 \frac{1}{x-1}\,dx &= \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1^+}\int_t^3 \frac{1}{x-1}\,dx \\ &= \lim_{t\to 1^-}[\ln|x-1|]_0^t + \lim_{t\to 1^+}[\ln|x-1|]_t^3 \\ &= \lim_{t\to 1^-}(\ln(1-t)-0) + \lim_{t\to 1^+}(\ln 2-\ln(t-1))\end{aligned}$$

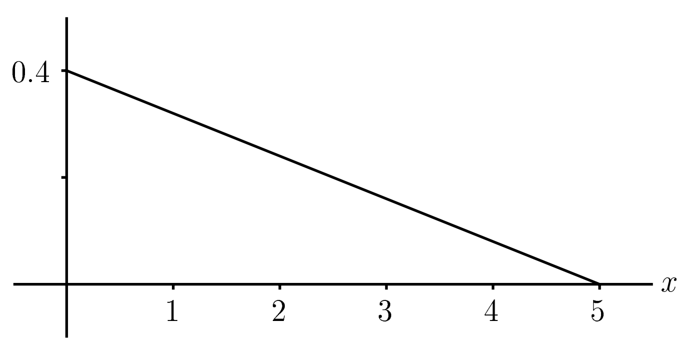
The first of these limits is infinite (since $\ln x \to -\infty$ as $x \to 0^+$), so the integral is divergent.

4. Suppose a random variable X has a probability density function f given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x<0 \\ \dfrac{10-2x}{25} & \text{if} \quad 0 \le x \le 5 \\ 0 & \text{if} \quad x>5 \end{cases}$$

Find the *median* value of X.

Solution: Here is a graph of the function $f(x)$. It is non-zero only on the interval $0 \le x \le 5$.



The median value of X is the number m between 0 and 5 such that

$$\int_0^m f(x)\,dx = \frac{1}{2}$$

To find m, we solve the equation

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{25}\int_0^m 10 - 2x\,dx \\
\frac{25}{2} &= [10x - x^2]_0^m \\
\frac{25}{2} &= 10m - m^2 \\
2m^2 - 20m + 25 &= 0
\end{aligned}$$

The quadratic formula says that $m = \dfrac{20 \pm \sqrt{200}}{4} = 5 \pm \dfrac{5\sqrt{2}}{2}$. Only the smaller of these numbers lies between 0 and 5, so the median must be

$$5 - \frac{5\sqrt{2}}{2}$$

Alternatively, we could find m using geometry. The vertical line $x = m$ cuts the region above into a trapezoid and a triangle. We want to find m so that each of these figures has area $\frac{1}{2}$. The area of the triangle, for example, would be

$$\frac{1}{2} \times b \times h = \frac{1}{2} \times (5 - m) \times \frac{10 - 2m}{25}$$

To find the median m, we set this area equal to $\frac{1}{2}$, getting the equation

$$\begin{aligned}
(5 - m) \times \frac{10 - 2m}{25} &= 1 \\
(5 - m)(10 - 2m) &= 25 \\
50 - 20m + 2m^2 &= 25 \\
2m^2 - 20m + 25 &= 0
\end{aligned}$$

This is the same quadratic we found above, and so of course the solution is the same.

5. Determine whether each of the following series converges or diverges. Give reasons.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$

Solution: Since this is a positive-term series and involves factorials, we are led to try the ratio test. Let

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

Then we have

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}{(n+1)!} \cdot \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n!}{(n+1)!} \\
&= \frac{2n+1}{n+1}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{2n+1}{n+1} \\
&= 2
\end{aligned}$$

Since $2 > 1$, we conclude that the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+5}}{n^2+4n}$

Solution: This is a positive-term series, and for large values of n, the terms are close to $\frac{\sqrt{n}}{n^2}$. We will apply the limit comparison test with $b_n = \frac{1}{n^{\frac{3}{2}}}$. Let a_n denote the n^{th} term of the given sequence. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n+5}}{n^2+4n} \cdot \frac{n^{\frac{3}{2}}}{1} \\
&= \lim_{n\to\infty} \sqrt{\frac{n+5}{n^4+8n^3+16n^2}} \sqrt{\frac{n^3}{1}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^3(n+5)}{n^4+8n^3+16n^2}} \\
&= \sqrt{\lim_{n\to\infty} \frac{n^4+5n^3}{n^4+8n^3+16n^2}} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

Since 1 is finite and non-zero, we know that the given series behaves as $\sum \frac{1}{n^{\frac{3}{2}}}$. This is a p-series with $p > 1$, so it converges. The given series also converges.

6. Find the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}(x+2)^n}{3^n\sqrt{n}}$

Solution: To begin, we apply the ratio test. Let $a_n = \dfrac{2^{n+1}(x+2)^n}{3^n\sqrt{n}}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{2^{n+2}|x+2|^{n+1}}{3^{n+1}\sqrt{n+1}} \cdot \frac{3^n\sqrt{n}}{2^{n+1}|x+2|^n} \\
&= \lim_{n\to\infty} \frac{2\sqrt{n}}{3\sqrt{n+2}}|x+2| \\
&= \frac{2}{3}|x+2| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \\
&= \frac{2}{3}|x+2|
\end{aligned}$$

The series converges absolutely for all values of x for which $\frac{2}{3}|x+2| < 1$. That is,

$$\begin{array}{rcccl}
-1 & < & \frac{2}{3}(x+2) & < & 1 \\
-\frac{3}{2} & < & x+2 & < & \frac{3}{2} \\
-\frac{7}{2} & < & x & < & -\frac{1}{2}
\end{array}$$

At the value $x = -\frac{7}{2}$, we have the series

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n\sqrt{n}}\left(-\frac{3}{2}\right)^n &= \sum_{n=1}^{\infty} (-1)^n \frac{2}{\sqrt{n}} \\
&= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}
\end{aligned}$$

This is an alternating series, and converges by the alternating series test.

At the value $x = -\dfrac{1}{2}$, we have the series

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n\sqrt{n}}\left(\frac{3}{2}\right)^n &= \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \\ &= 2\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\end{aligned}$$

This is a p-series with $p < 1$, so it diverges.

The interval of convergence is $\left[-\dfrac{7}{2}, -\dfrac{1}{2}\right)$.

7. Find the sums of the series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{3n+1}}{3^{2n}}$

   Solution: We rewrite the series as

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot (2^3)^n}{(3^2)^n} &= \sum_{n=0}^{\infty} 2\left(\frac{-1 \cdot 8}{9}\right)^n \\ &= \sum_{n=0}^{\infty} 2\left(-\frac{8}{9}\right)^n\end{aligned}$$

   This is a geometric series, and since $\left|-\dfrac{8}{9}\right| < 1$, we know that its sum is

$$\begin{aligned}\frac{2}{1-(-\frac{8}{9})} &= \frac{2}{\frac{17}{9}} \\ &= \frac{18}{17}\end{aligned}$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{9^n}$

   Solution: This series has the form $\displaystyle\sum_{n=1}^{\infty} nx^n$, where $x = \dfrac{2}{9}$. We know that for $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

We differentiate both sides of this equation to get

$$\begin{aligned}\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} \\ &= \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

Next we multiply both sides by x to get

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

The given series thus sums to

$$\begin{aligned}\frac{\frac{2}{9}}{\left(1-\frac{2}{9}\right)^2} &= \frac{2}{9} \times \frac{81}{49} \\ &= \frac{18}{49}\end{aligned}$$

8. Use a power series to estimate the value of $\int_0^{\frac{1}{4}} \frac{1}{1+x^3}\, dx$ with an error of less than 10^{-6}.

Solution: We begin with the fact that

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

for $|x| < 1$. Substituting $-x^3$ for t, we get

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

for $|x| < 1$. We integrate both sides from 0 to $\frac{1}{4}$ to get

$$\begin{aligned}\int_0^{\frac{1}{4}} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1}\Bigg]_0^{\frac{1}{4}} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(3n+1)4^{3n+1}}\end{aligned}$$

Since this series satisfies the hypotheses of the alternating series test, we know that the error in any partial sum is less than the abosolute value of the first term we omit. By trial and error, we find that the $n = 3$ term has absolute value less than 10^{-6}. We can estimate the integral to the desired accuracy using just the $n = 0$, 1, and 2 terms. We have

$$\begin{aligned}
\int_0^{\frac{1}{4}} \frac{1}{1+x^3}\,dx &\approx \frac{1}{4} - \frac{1}{4\times 4^4} + \frac{1}{7\times 4^7} \\
&\approx 0.249032
\end{aligned}$$

9. Let $f(x) = x^{-\frac{1}{3}}$. Find the fourth-order Taylor polynomial for $f(x)$ about $a = 1$.

We have

$$\begin{aligned}
f(x) &= x^{-\frac{1}{3}} \\
f'(x) &= -\frac{1}{3}x^{-\frac{4}{3}} \\
f''(x) &= \frac{4}{9}x^{-\frac{7}{3}} \\
f'''(x) &= -\frac{28}{27}x^{-\frac{10}{3}} \\
f^{(4)}(x) &= \frac{280}{81}x^{-\frac{13}{3}}
\end{aligned}$$

so that

$$\begin{aligned}
f(1) &= 1 \\
f'(1) &= -\frac{1}{3} \\
f''(1) &= \frac{4}{9} \\
f'''(1) &= -\frac{28}{27} \\
f^{(4)}(x) &= \frac{280}{81}
\end{aligned}$$

The fourth-order Taylor polynomial $T_4(x)$ is given by

$$T_4(x) = 1 - \frac{1}{3}(x-1) + \frac{4}{9}\frac{(x-1)^2}{2!} - \frac{28}{27}\frac{(x-1)^3}{3!} + \frac{280}{81}\frac{(x-1)^4}{4!}$$