Math 101 **Logarithms Worksheet** **September 2003**

Definition: For $a > 0$, $a \neq 1$ and $x > 0$, we have

$$\boxed{y = \log_a x \text{ if and only if } a^y = x.}$$

More informally, $\log_a x$ is *the power you raise a to to get x.*

Example: Evaluate $\log_3 \frac{1}{9}$.

Solution: We note that $\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$. The power you raise 3 to to get $\frac{1}{9}$ is -2, so we have

$$\log_3 \frac{1}{9} = -2.$$

Definition: We write $\ln x$ for $\log_e x$, so that $\ln x$ is *the power you raise e to to get x.* This is called the *natural logarithm* function.

Exercises I: Evaluate

a) $\log_2 16$

b) $\log_4 \frac{1}{2}$

c) $\log_5 125$

d) $\log_7 1$

e) $\log_5 5\sqrt{5}$

f) $\log_{10} 100$

g) $\log_{100} 10$

h) $\ln \frac{1}{\sqrt{e}}$

Cancellation Laws: From the definition of the logarithm, we get the identities

$$\boxed{\begin{array}{rclcrcl} \log_a a^x & = & x & \qquad & \ln e^x & = & x \\ a^{\log_a x} & = & x \text{ (for } x > 0) & \qquad & e^{\ln x} & = & x \text{ (for } x > 0) \end{array}}$$

We can use these identities to solve equations involving exponents and logarithms.

Example: Solve the equation $3^{2x-2} = 27$ for x.
Solution: We take logarithms to the base 3, getting

$$\log_3 3^{2x-2} = \log_3 27.$$

By a cancellation law, the left side is equal to $2x - 2$. The right side is equal to 3, since $3^3 = 27$. Thus we have

$$2x - 2 = 3.$$

We solve this to get $x = \dfrac{5}{2}$.

Example: Solve the equation $\ln(7 - 2x) = 3$ for x.
Solution: From the given equation, we get

$$e^{\ln(7-2x)} = e^3.$$

We apply a cancellation law to the left side to get

$$7 - 2x = e^3.$$

We solve this for x, getting $x = \dfrac{7 - e^3}{2}$. This is the exact form of the answer; if necessary, we can use a calculator to get the approximation $x \approx -6.543$.

Exercises II: Solve for x:

a) $\log_2 x = -1$

b) $\log_5(1 - 4x) = \dfrac{1}{2}$

c) $\ln(3x + 5) = -2$

d) $4^{x-1} = \dfrac{1}{16}$

e) $3^{1-2x} - \dfrac{\sqrt{3}}{3} = 0$

f) $e^{2x-5} = 8$

Log Laws: The following identities relate sums, differences, and multiples of logarithms to logarithms of products, quotients, and powers. As usual, we must have $a > 0$ and $a \neq 1$, and any number whose logarithm we take must be positive.

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y & \qquad \ln(xy) &= \ln x + \ln y \\ \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y & \qquad \ln\left(\frac{x}{y}\right) &= \ln x - \ln y \\ \log_a(x^r) &= r \log_a x & \qquad \ln(x^r) &= r \ln x \end{aligned}$$

Example: Rewrite the expression $\log_3 45 - \log_3 5$ as a single logarithm, and simplify it if possible.
Solution: By the second log law, we get

$$\begin{aligned}\log_3 45 - \log_3 5 &= \log_3\left(\frac{45}{5}\right)\\ &= \log_3 9\end{aligned}$$

and since $3^2 = 9$, we know that this expression is equal to 2.
Example: Solve the equation $\ln x + \ln(x+2) - 1 = 0$ for x.
Solution: We use the first log law (for ln) to combine the two logarithms, getting

$$\begin{aligned}\ln(x(x+2)) - 1 = 0&\\ \ln(x^2+2x) &= 1.\end{aligned}$$

Next we exponentiate both sides (that is, raise e to both sides) to get

$$\begin{aligned}x^2 + 2x &= e\\ x^2 + 2x - e &= 0.\end{aligned}$$

We apply the quadratic formula to find two solutions

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4+4e}}{2}\\ &= -1 \pm \sqrt{1+e}\end{aligned}$$

In checking these answers, we observe that the root $-1 - \sqrt{1+e}$ is negative, and so $\ln(-1 - \sqrt{1+e})$ is undefined. We must therefore reject this solution. The other root, $x = -1 + \sqrt{1+e}$, does satisfy the given equation.

Exercises III: Write as a single logarithm, and simplify if possible.

a) $\log_2 10 + \log_2 12.8$

b) $\log_{10} 560 - \log_{10} 5.6$

c) $\frac{1}{2}\log_5 8 - \log_5 2$

d) $\ln x - \frac{1}{2}\ln(x^2+1)$

Solve for x:

a) $\log_3 x + \log_3 7 = 2$

b) $\ln x - \ln(x^2 - 1) = 2$

Change of base Formula: For $a > 0$, $a \neq 1$, and $x > 0$, we have the identity

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

We can use this to convert any logarithmic expression into an expression involving only the natural logarithm, ln. Most calculators have a natural logarithm key, and the change of base formula gives us a handy way to approximate logarithms with arbitrary bases on a calculator.

Example: Use a calculator to approximate $\log_3 7$.
Solution: From the change of base formula, we get

$$\log_3 7 = \frac{\ln 7}{\ln 3}$$

The calculator says that this is approximately 1.77124.
Example: Solve the equation $3^{12x-4} = 19$ for x, giving the answer in terms of natural logarithms.
Solution: We take logarithms to the base 3 to get

$$12x - 4 = \log_3 19$$

and solve for x, getting $x = \dfrac{\log_3 19 + 4}{12} = \dfrac{1}{12}\log_3 19 + \dfrac{1}{3}$. We use the change of base formula to put the answer in the required form,

$$x = \frac{\ln 19}{12 \ln 3} + \frac{1}{3}.$$

Alternate solution: We begin by taking natural logarithms, getting

$$\ln 3^{12x-4} = \ln 19$$

and then apply a log law to get

$$\begin{aligned} (12x-4)\ln 3 &= \ln 19 \\ 12x - 4 &= \frac{\ln 19}{\ln 3} \\ 12x &= \frac{\ln 19}{\ln 3} + 4 \\ x &= \frac{\ln 19}{12 \ln 3} + \frac{1}{3}. \end{aligned}$$

Exercises IV: Solve each equation for x. Give the answer first in terms of natural logarithms, and then as a decimal approximation.

a) $2^{1-4x} = 7$

b) $10e^{1.5x} = 1500$

c) $3^x = 7 \cdot 9^{2x-5}$

d) $1.06^{\frac{x}{12}} = 2$