1. Let $f(x) = e^{-x^2}$.

   (a) Use the midpoint rule to estimate $\int_0^2 f(x)\,dx$ with $n = 20$ and $n = 50$ subintervals.
   Solution: With $n = 20$ subintervals in the interval $[0, 2]$, we have $\Delta x = \frac{2-0}{20} = \frac{1}{10}$. The twenty subintervals are

   $$\left[0, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{2}{10}\right], \ldots, \left[\frac{19}{10}, 2\right]$$

   and their midpoints are

   $$\frac{1}{20}, \frac{3}{20}, \ldots, \frac{39}{20}$$

   In the language of the calculator, M_{20} is

   ```
   (1/10)sum seq(y1,x,1/20,39/20,1/10)
   ```

   where `y1` has been set to `e^(-x^2)`. The value is about 0.8821118278.
   With $n = 50$ subintervals in the interval $[0, 2]$, we have $\Delta x = \frac{2-0}{50} = \frac{1}{25}$. The twenty subintervals are

   $$\left[0, \frac{1}{25}\right], \left[\frac{1}{25}, \frac{2}{25}\right], \ldots, \left[\frac{49}{25}, 2\right]$$

   and their midpoints are

   $$\frac{1}{50}, \frac{3}{50}, \ldots, \frac{99}{50}$$

   We calculate

   ```
   (1/25)sum seq(y1,x,1/50,99/50,1/25)
   ```

   and get $M_{50} \approx 0.8820862727$.

   (b) Use the trapezoid rule to estimate $\int_0^2 f(x)\,dx$ with $n = 20$ and $n = 50$ subintervals.
   Solution: As above, the subintervals for $n = 20$ are

   $$\left[0, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{2}{10}\right], \ldots, \left[\frac{19}{10}, 2\right]$$

so the expression we want to evaluate is

$$\frac{1}{20}\left[f(0)+2f\left(\frac{1}{10}\right)+2f\left(\frac{2}{10}\right)+\cdots+2f\left(\frac{19}{10}\right)+f(2)\right]$$

The calculator says this is approximately 0.8820204404.

For $n = 50$, the subintervals are again

$$\left[0,\frac{1}{25}\right],\ \left[\frac{1}{25},\frac{2}{25}\right],\ \ldots,\ \left[\frac{49}{25},2\right]$$

so we want to evaluate

$$\frac{1}{50}\left[f(0)+2f\left(\frac{1}{25}\right)+2f\left(\frac{2}{25}\right)+\cdots+2f\left(\frac{49}{25}\right)+f(2)\right]$$

The calculator says that $T_{50} \approx 0.8820716250$.

(c) Find the maximum value of $|f''(x)|$ on the interval $[0, 2]$.

Solution: We have

$$\begin{aligned} f'(x) &= -2xe^{-x^2} \\ f''(x) &= (-2x)(-2x)e^{-x^2} + (-2)e^{-x^2} \\ &= (4x^2-2)e^{-x^2} \end{aligned}$$

We graph f'' on the calculator, and note that the maximum value of $|f''(x)|$ appears to be $|f''(0)|$, when $f''(x)$ has its minimum on the given interval. Since $f''(0) = -2$, we take 2 for the maximum value of $|f''(x)|$ on the interval $[0, 2]$

(d) Find the error estimates for M_{50} and T_{50}. Use these to give an interval inside which the exact value of $\int_0^2 f(x)\,dx$ must lie.

Solution: Using the formulas in the book with $K_2 = 2$, $n = 50$, $a = 0$, and $b = 2$, we get

$$\begin{aligned} E_M &= \frac{2(2)^3}{24(50)^2} \\ &= \frac{1}{3750} \\ &\approx 0.00026667 \end{aligned}$$

Then E_T is just twice E_M, so we get

$$E_T = \frac{1}{1875} \approx 0.00053333$$

We know that the actual value of the integral must lie between $M_{50} - E_M$ and $M_{50} + E_M$, that is, between

$$0.88181960 \quad \text{and} \quad 0.88235294$$

(Note that we rounded the lower bound downward and the upper bound upward, just in case.)
From the trapezoid calculation, we know that the actual value of the integral must lie between

$$0.88153829 \quad \text{and} \quad 0.88260496$$

so we don't get any new information from the trapezoid rule in this case.

2. Repeat the steps above with $f(x) = \cos(x^2)$.

Solution: We get

$$\begin{aligned} M_{20} &= \frac{1}{10}\left[f\left(\frac{1}{20}\right) + f\left(\frac{3}{20}\right) + \cdots + f\left(\frac{39}{20}\right)\right] \\ &\approx 0.4601961532 \end{aligned}$$

and

$$\begin{aligned} M_{50} &= \frac{1}{25}\left[f\left(\frac{1}{50}\right) + f\left(\frac{3}{50}\right) + \cdots + f\left(\frac{99}{50}\right)\right] \\ &\approx 0.4612595466 \end{aligned}$$

For the trapezoid rule, we get

$$\begin{aligned} T_{20} &= \frac{1}{20}\left[f(0) + 2f\left(\frac{1}{10}\right) + 2f\left(\frac{2}{10}\right) + \cdots + 2f\left(\frac{19}{10}\right) + f(2)\right] \\ &\approx 0.4639886774 \end{aligned}$$

and

$$\begin{aligned} T_{50} &= \frac{1}{50}\left[f(0) + 2f\left(\frac{1}{25}\right) + 2f\left(\frac{2}{25}\right) + \cdots + 2f\left(\frac{49}{25}\right) + f(2)\right] \\ &\approx 0.4618652068 \end{aligned}$$

We find that $f''(x) = -4x^2\cos(x^2) - 2\sin(x^2)$. From a calculator graph, it appears that the maximum of $|f''(x)|$ on $[0, 2]$ occurs when $x \approx 1.882$ and the maximum value

is about 13.827. Just to be safe, we'll take $K_2 = 14$, which seems like a reasonable upper bound for $|f''(x)|$ on $[0,2]$. (Any number bigger than the maximum value of $f''(x)$ will do.)

With this value for K_2, and with $n = 50$, we get

$$E_M = \frac{14(2^3)}{24(50)^2} = \frac{7}{3750} \approx 0.00186667$$

and

$$E_T = 2E_M = \frac{7}{1875} \approx 0.00373333$$

Using the midpoint rule approximation and its error bounds, we find that the exact value of our integral must lie between

$$0.45939287 \quad \text{and} \quad 0.46312622$$

(where again we have rounded the lower bound down and the upper bound up). Using the trapezoid rule approximation and its error bounds, we find that the exact value of the integral must lie between

$$0.45813187 \quad \text{and} \quad 0.46559855$$